Una nota sullo sviluppo della derivata di ordine n (n intero positivo)

delle funzioni trigonometriche P(x) = Tan(x), e C(x) = Sec(x).

Considerazioni ed osservazioni di Pasquale Cutolo – (p.cutolo@inwind.it)

Development of derivatives of order n (n positive integer) of the trigonometric

functions P(x) = Tan(x) and C(x) = Sec(x). Some remarks by

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Introduzione-

Con il presente studio l'autore, utilizzando formule note, esamina lo sviluppo delle derivate, di ordine (2n-1), e di ordine (2n), (n = 1,2,3,…), delle funzioni trigonometriche P(x) = Tan(x), e C(x) = Sec(x), determinando le espressioni che definiscono i coefficienti, in funzione di n, dei polinomi risultanti, ottenendo formule complesse straordinarie ed inusuali. Alla fine viene aggiunta una interessante applicazione.

Abstract :

In this report the author uses well known formulas to revisit the development of the derivatives of order (2n-1) and (2n), (n = 1,2,3,…), of the trigonometric functions P(x) = Tan(x) and C(x) = Sec(x).

The formulas for the functions of n giving the coefficients of the resulting polynomial expansions are derived through unusual developments of impressive complexity. In the conclusions an interesting application is presented.

A.0.0- Caso A – P(x) = Tan(x)

E' nota la seguente relazione dei complementi:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{Sin(\pi z)}, \qquad 0 < z < 1 \qquad \text{(A.0.0.01)}$$

essendo, $\Gamma(z)=\int_0^\infty t^{z-1}e^{-z}$, $\text{Re}(z)>0$, la ben nota funzione di Eulero di seconda specie.

Ponendo, nella (A.0.0.01), $z=\frac{1}{2}-x$, (x reale), e prendendo il logaritmo naturale di ambo i membri della (A.0.0.01), otteniamo:

$$\ln\Gamma(\frac{1}{2}-x)+\ln\Gamma(\frac{1}{2}+x) = \ln\pi-\ln Cos(\pi x) \qquad \text{(A.0.0.02)}$$

Derivando, rispetto ad x, i due membri della (A.0.0.02), ricaviamo:

$$\Psi(\frac{1}{2}+x)-\Psi(\frac{1}{2}-x)=\pi Tan(\pi x), \qquad \text{(A.0.0.03)}$$

essendo $\Psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, e $Tan(\pi x)$ è la tangente trigonometrica di (πx).

Ricordando che $\Psi(x)=\int_0^1 \frac{t^{x-1}-1}{t-1}dt+\gamma$, (Ved. [1] , 9.541, pag. 1074), essendo γ = 0,5772156649 la costante di Eulero-Mascheroni, dalla (A.0.0.03) ricaviamo:

$$\int_0^1 \frac{t^{x-1/2}-t^{-x-1/2}}{t-1}dt=(t=e^{-u})=\int_0^\infty \frac{e^{-u(x-1/2)}-e^{-u(-x-1/2)}}{e^{-u}-1}e^{-u}du=\int_0^\infty \frac{(e^{ux}-e^{-ux})e^{u/2}}{e^{u}-1}du =$$

$$=\int_0^\infty \frac{Sinh(ux)}{Sinh(u/2)}du = \pi Tan(\pi x), \qquad |x|<\frac{1}{2}, \qquad \text{(A.0.0.04)}$$

A.1.0- **Derivata di ordine** (2n-1) della funzione P(x) = $Tan(x)$.

Derivando, (2n-1) volte, la (A.0.0.04), rispetto ad x, troviamo:

$$\int_0^\infty \frac{u^{2n-1}Cosh(ux)}{Sinh(u/2)}du = [\pi Tan(\pi x)]^{(2n-1)} = \pi^{2n}\sum_{h=0}^{n} a_h[Tan(\pi x)]^{2h}$$

$$=\frac{-2\pi}{i}[\sum_{k\geq 0}(-1)^k e^{-2\pi i x(1+k)}]^{(2n-1)} \qquad \text{(A.1.0.01)}$$

Non è difficile verificare la relazione:

$$[\pi Tan(\pi x)]^{(2n-1)} = \pi^{2n}\sum_{h=0}^{n} a_h[Tan(\pi x)]^{2h} \qquad \text{(A.1.0.02)}$$

e la relazione $\qquad [Tan(x)]^{(2n-1)} = \sum_{h=0}^{n} a_h[Tan(x)]^{2h}$

A.1.1 - **Determinazione del termine noto** a_0 del polinomio rappresentato dallo sviluppo della funzione $[Tan(x)]^{(2n-1)} = \sum_{h=0}^{n} a_h[Tan(x)]^{2h}$

Dalla (A.1.0.01) ricaviamo:

$$[\pi Tan(\pi x)]^{(2n-1)} = [\frac{\pi}{i}(1-\frac{2}{e^{2\pi i x}+1})]^{(2n-1)} = \frac{-2\pi}{i}[\sum_{k\geq 0}(-1)^k e^{-2\pi i x(1+k)}]^{(2n-1)} =$$

$$=\frac{-2\pi}{i}\sum_{k\geq 0}(-1)^k[-2\pi i(1+k)]^{2n-1}e^{-2\pi i x(1+k)}, \qquad \text{(A.1.1.01)}$$

$i = \sqrt{-1}$; sostituendo k a (1+k), abbiamo:

$$[\pi Tan(\pi x)]^{(2n-1)} \; = \; 2^{2n}\pi^{2n}(-1)^n \sum_{k\geq 1}(-1)^k k^{2n-1} e^{-2\pi ixk} \qquad \text{(A.1.1.02)}$$

Ponendo, nella (A.1.0.01) e (A.1.1.02), x = 0, troviamo:

$$\int_0^\infty \frac{u^{2n-1}}{Sinh(u/2)} du \; = \; \pi^{2n} a_0 \; = \; 2^{2n}\pi^{2n}(-1)^n \sum_{k\geq 1}(-1)^k k^{2n-1} \qquad \text{(A.1.1.03)}$$

Osserviamo che:

$$\sum_{k\geq 1}(-1)^k k^{2n-1} = \sum_{k\geq 1}(2k)^{2n-1} - \sum_{k\geq 1}(2k-1)^{2n-1} =$$

$$\sum_{k\geq 1}(2k)^{2n-1} - [\sum_{k\geq 1}(2k-1)^{2n-1} + \sum_{k\geq 1}(2k)^{2n-1} - \sum_{k\geq 1}(2k)^{2n-1}] =$$

$$= \sum_{k\geq 1}(2k)^{2n-1} - \sum_{k\geq 1}k^{2n-1} + \sum_{k\geq 1}(2k)^{2n-1} = (2^{2n}-1)\zeta(1-2n) \, ; \qquad \text{(A.1.1.04)}$$

ricordando che $\zeta(1-2n) = -\frac{B_{2n}}{2n}$, e $\zeta(-2n) = 0$, (Ved. $|1|$, 9.541, pag. 1074),

troviamo che: $\sum_{k\geq 1}(-1)^k k^{2n-1} = (2^{2n}-1)(-\frac{B_{2n}}{2n})$ (A.1.1.05)

essendo B_{2n} il numero di Bernoulli di indice 2n, mentre $\zeta(s)$ è la funzione

Zeta di Riemann, definita da: $\zeta(s) = \sum_{k\geq 1}\frac{1}{k^s}$, Re(s) > 1

Sostituendo il risultato della (A.1.1.05) nella (A.1.1.03), otteniamo:

$$\pi^{2n} a_0 \; = \; 2^{2n}\pi^{2n}(-1)^{n-1}(2^{2n}-1)\frac{B_{2n}}{2n}$$

Ricordando, inoltre, che $(-1)^{n-1}B_{2n} = |B_{2n}|$, e semplificando, abbiamo:

$$a_0 \; = \; 2^{2n}(-1)^{n-1}(2^{2n}-1)\frac{B_{2n}}{2n} \; = \; 2^{2n}(2^{2n}-1)\frac{|B_{2n}|}{2n} \qquad \text{(A.1.1.06)}$$

che rappresenta l'espressione del termine noto, in funzione di n, del polinomio

$\sum_{h=0}^{n} a_h [Tan(x)]^{2h}$, di grado 2n.

Osserviamo che, per n intero positivo, a_0 risulta sempre intero positivo;
poiché $|B_{2n}|$ è dato dal rapporto di due numeri interi, indicando con p e q
detti numeri interi, ($|B_{2n}|$ = p/q), l'espressione (A.1.1.06) indica che a_0 rappresenta
un numero che è un multiplo intero del numero intero q e di 2n.
L'espressione di a_0 possiamo ricavarla anche utilizzando la relazione (A.1.1.03), infatti:

$$\int_0^\infty \frac{u^{2n-1}}{Sinh(u/2)} du \; = \; \pi^{2n} a_0 \; = \; 2\int_0^\infty u^{2n-1} e^{-u/2} \sum_{k\geq 0} e^{-uk} du \; = \; 2\sum_{k\geq 0}\frac{(2n-1)!}{(\frac{1}{2}+k)^{2n}} =$$

$$= 2^{2n+1}(2n-1)! \sum_{k\geq 0}\frac{1}{(1+2k)^{2n}} \; ; \text{ ma,}$$

$$\sum_{k\geq 0}\frac{1}{(1+2k)^{2n}} = \sum_{k\geq 0}\frac{1}{(1+2k)^{2n}} + \sum_{k\geq 1}\frac{1}{(2k)^{2n}} - \sum_{k\geq 1}\frac{1}{(2k)^{2n}} =$$

$$=\sum_{k\ge1}\frac{1}{(k)^{2n}}-\sum_{k\ge1}\frac{1}{(2k)^{2n}} = (1-2^{-2n})\zeta(2n)\,; \qquad (A.1.1.07)$$

ricordando che: $\zeta(2n) = \dfrac{2^{2n-1}\pi^{2n}}{(2n)!}\left|B_{2n}\right|,$ (A.1.1.08)

(Ved. [1], 9.541, pag. 1074), otteniamo:

$$2^{2n+1}(2n-1)!\sum_{k\ge0}\frac{1}{(1+2k)^{2n}}=2^{2n+1}(2n-1)!(1-2^{-2n})\,\frac{2^{2n-1}\pi^{2n}}{(2n)!}\left|B_{2n}\right|=\pi^{2n}a_0\,, \qquad (A.1.1.09)$$

da cui: $a_0 = 2^{2n}(2^{2n}-1)\dfrac{\left|B_{2n}\right|}{2n}$

La relazione (A.1.1.06) è stata verificata con un programma di matematica.
Riportiamo i valori di a_0 per n variabile da 1 a 10.

n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
a_0 = 1, 2, 16, 272, 7936, 353792, 22368256, 1903757312, 209865342976, 29088885112832

Dalla (A.1.1.09) ricaviamo:

$$\sum_{k\ge0}\frac{1}{(1+2k)^{2n}} = \frac{(2^{2n}-1)\pi^{2n}}{2(2n)!}\left|B_{2n}\right| \qquad (A.1.1.10)$$

A.1.2. – Determinazione dell'espressione della Somma $\sum_{h=0}^{n}a_h$, in funzione di n.

Utilizzando le relazioni (A.1.0.01) e (A.1.1.02), e ponendo, x = ¼, troviamo:

$$\pi^{2n}\sum_{h=0}^{n}a_h = 2^{2n}\pi^{2n}(-1)^n\sum_{k\ge1}(-1)^k k^{2n-1}e^{-2\pi ik/4} = 2^{2n}\pi^{2n}(-1)^n\sum_{k\ge1}(-1)^k k^{2n-1}(-i)^k\text{ , cioè:}$$

$$\sum_{h=0}^{n}a_h = 2^{2n}(-1)^n\sum_{k\ge1}(i)^k k^{2n-1} = 2^{2n}(-1)^n[\sum_{k\ge1}(-1)^k(2k)^{2n-1}+\sum_{k\ge1}(i)^{2k-1}(2k-1)^{2n-1}];$$

uguagliando le parti reali della precedente relazione, troviamo:

$$\sum_{h=0}^{n}a_h = 2^{2n}(-1)^n\sum_{k\ge1}(-1)^k(2k)^{2n-1}\ ;$$

utilizzando la (A.1.1.04), o la (A.1.1.05), abbiamo:

$$\sum_{h=0}^{n}a_h = 2^{2n}2^{2n-1}(-1)^n(2^{2n}-1)\zeta(1-2n) = 2^{4n-1}(-1)^{n-1}(2^{2n}-1)\frac{B_{2n}}{2n} =$$

$$= 2^{4n-1}(2^{2n}-1)\frac{\left|B_{2n}\right|}{2n}, \qquad (A.1.2.01)$$

La relazione (A.1.2.01) è stata verificata con un programma di matematica.
Riportiamo i primi valori di $\sum_{h=0}^{n}a_h$ per n variabile da 1 a 10.

n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

$\sum_{h=0}^{n}a_h$ = 2, 16, 512, 34816, 4063232, 724566016, 183240753152, 62382319599616,
27507470234550272, 15250953398036463616

Dividendo il risultato della (A.1.2.01) con quello della (A.1.1.06), otteniamo:

$$(\sum_{h=0}^{n} a_h)/a_0 = 2^{2n-1} \qquad \text{(A.1.2.02)}$$

A.1.3 -Determinazione delle espressioni che definiscono, in funzione di n, i coefficienti $a_{1,} a_{2,} a_{3,} ..., a_n$ dei termini del polinomio $\sum_{h=0}^{n} a_h [Tan(x)]^{2h}$

Riprendiamo la formula indicata con (A.1.1.02)

$$[\pi Tan(\pi x)]^{(2n-1)} = \pi^{2n} \sum_{h=0}^{n} a_h [Tan(\pi x)]^{2h} = 2^{2n} \pi^{2n} (-1)^n \sum_{k\geq 1} (-1)^k k^{2n-1} e^{-2\pi i x k} , \qquad \text{(A.1.3.01)}$$

e poniamo $Tan(\pi x) = t$, da cui: $\pi x = ArcTan(t) = \frac{1}{2i} \ln \frac{1+it}{1-it}$,

che sostituita nella (A.1.3.01) fornisce:

$$\sum_{h=0}^{n} a_h t^{2h} = 2^{2n} (-1)^n \sum_{k\geq 1} (-1)^k k^{2n-1} e^{-k \ln\frac{1+it}{1-it}} = 2^{2n} (-1)^n \sum_{k\geq 1} (-1)^k k^{2n-1} (\frac{1-it}{1+it})^k ; \qquad \text{(A.1.3.02)}$$

derivando la (A.1.3.02), (2h) volte, rispetto a t, e ponendo dopo, t = 0, abbiamo:

$$a_h (2h)! = [2^{2n} (-1)^n \sum_{k\geq 1} (-1)^k k^{2n-1} (\frac{1-it}{1+it})^k]_{t=0}^{(2h)} ;$$

ora, $[(1-it)^k (1+it)^{-k}]_{t=0}^{(2h)} = \lim_{t \to 0} \sum_{j=0}^{2h} \binom{2h}{j} [(1-it)^k]^{(2h-j)} [(1+it)^{-k}]^{(j)} =$

$$= \sum_{j=0}^{2h} \binom{2h}{j} \frac{\Gamma(k+1)(-i)^{2h-j}}{\Gamma(k+1-2h+j)} \frac{\Gamma(k+j)(-i)^j}{\Gamma(k)} = \sum_{j=0}^{2h} \binom{2h}{j} k(-1)^h \frac{\Gamma(k+j)}{\Gamma(k+1-2h+j)} =$$

$$= \sum_{j=0}^{2h} \binom{2h}{j} k(-1)^h (k+j-1)(k+j-2)...(k+j-2h+1) ;$$

ricordiamo che nello sviluppo delle derivate abbiamo utilizzate le formule seguenti:

$$D_x^{(n)} (a+bx)^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-n)} b^n (a+bx)^{\alpha-n} ;\ D_x^{(n)} (a+bx)^{-\beta} = \frac{\Gamma(n+\beta)}{\Gamma(\beta)} (-b)^n (a+bx)^{-\beta-n} ,$$

(Ved. [5]);

inoltre, ricordiamo la ben nota relazione:

$$x(x-1)(x-2)\ldots(x-n+1) = \sum_{k=0}^{n} s(n,k) x^k , \qquad \text{(A.1.3.03)}$$

dove i coefficienti s(n,k) rappresentano i numeri di Stirling di prima specie.
(Ved. [8])

Pertanto: $\sum_{j=0}^{2h} \binom{2h}{j} k(-1)^h (k+j-1)(k+j-2)...(k+j-2h+1) =$

$$= \sum_{j=0}^{2h} \binom{2h}{j} k(-1)^h \sum_{u=0}^{2h-1} s(2h-1,u)(k+j-1)^u = \sum_{j=0}^{2h} \binom{2h}{j} k(-1)^h \sum_{u=0}^{2h-1} s(2h-1,u) \sum_{p\geq 0} \binom{u}{p} k^p (j-1)^{u-p} ;$$

quindi: $a_h (2h)! = 2^{2n} (-1)^n \sum_{k\geq 1} (-1)^k k^{2n} \sum_{j=0}^{2h} \binom{2h}{j} (-1)^h \sum_{u=0}^{2h-1} s(2h-1,u) \sum_{p\geq 0} \binom{u}{p} k^p (j-1)^{u-p} =$

$$=2^{2n}(-1)^n(-1)^h\sum_{j=0}^{2h}\binom{2h}{j}\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{p\geq 0}\binom{u}{p}(j-1)^{u-p}\sum_{k\geq 1}(-1)^k k^{2n+p};$$

sappiamo che $\zeta(-2n-2p)=0$, [(n + p) intero > 0)], e quindi:

$$a_h(2h)! = 2^{2n}(-1)^n(-1)^h\sum_{j=0}^{2h}\binom{2h}{j}\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{p\geq 1}\binom{u}{2p-1}(j-1)^{u-2p+1}\sum_{k\geq 1}(-1)^k k^{2n+2p-1};$$

ricordiamo che:

$$\sum_{k\geq 1}(-1)^k k^{2n+2p-1} = (2^{2n+2p}-1)\zeta(1-2n-2p) = (2^{2n+2p}-1)(-\frac{B_{2n+2p}}{2n+2p});$$

quindi: $a_h(2h)! =$

$$=2^{2n}(-1)^{n-1}(-1)^h\sum_{j=0}^{2h}\binom{2h}{j}\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{p\geq 1}\binom{u}{2p-1}(j-1)^{u-2p+1}(2^{2n+2p}-1)\frac{B_{2n+2p}}{2n+2p}; \quad \text{(A.1.3.04)}$$

osserviamo che nello sviluppo della relazione precedente (A.1.3.04), al variare di j, u, p, si presentano valori indeterminati (0^0). Per eliminare detti inconvenienti calcoliamo l'espressione (A.1.3.04), per j = 0, per j = 1, e per j > 1.

Così operando, troviamo:

1) per j = 0, $\sum_{j=0}^{2h}\binom{2h}{j}k(-1)^h(k+j-1)(k+j-2)...(k+j-2h+1)=$

$$=(-1)^h k(k-1)(k-2)...(k-2h+1)=(-1)^h\sum_{u=0}^{2h}s(2h,u)k^u;$$

$$[a_h(2h)!]_{j=0}=2^{2n}(-1)^n\sum_{k\geq 1}(-1)^k k^{2n-1}\ (-1)^h\sum_{u=0}^{2h}s(2h,u)k^u=$$

$$=2^{2n}(-1)^n(-1)^h\sum_{u=0}^{2h}s(2h,u)\sum_{k\geq 1}(-1)^k k^{2n+u-1}=$$

$$=2^{2n}(-1)^n(-1)^h\sum_{u=0}^{h}s(2h,2u)\sum_{k\geq 1}(-1)^k k^{2n+2u-1}=$$

$$=2^{2n}(-1)^n(-1)^h\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u}-1)\zeta(1-2n-2u)=$$

$$=2^{2n}(-1)^{n-1}(-1)^h\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u}$$

2) per j = 1, $\sum_{j=0}^{2h}\binom{2h}{j}k(-1)^h(k+j-1)(k+j-2)...(k+j-2h+1)=$

$$=2hk(-1)^h(k)(k-1)...(k-2h+2)= 2hk(-1)^h\sum_{u=0}^{2h-1}s(2h-1,u)k^u;$$

$$a_h(2h)!]_{j=1} = 2^{2n}(-1)^n\sum_{k\geq 1}(-1)^k k^{2n-1}\ 2hk(-1)^h\sum_{u=0}^{2h-1}s(2h-1,u)k^u=$$

$$=2^{2n+1}(-1)^n h(-1)^h\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{k\geq 1}(-1)^k k^{2n+u}=$$

$$=2^{2n+1}(-1)^n h(-1)^h\sum_{u=1}^{h}s(2h-1,2u-1)\sum_{k\geq 1}(-1)^k k^{2n+2u-1}; \text{ pertanto:}$$

$$a_h(2h)! = 2^{2n}(-1)^{n-1}(-1)^h\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u}+$$

$$+2^{2n+1}(-1)^n h(-1)^h\sum_{u=1}^{h} s(2h-1,2u-1)\sum_{k\geq 1}(-1)^k k^{2n+2u-1}+$$

$$+2^{2n}(-1)^{n-1}(-1)^h\sum_{j=2}^{2h}\binom{2h}{j}\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{p\geq 1}\binom{u}{2p-1}(j-1)^{u-2p+1}(2^{2n+2p}-1)\frac{B_{2n+2p}}{2n+2p}.$$

In definitiva, ricaviamo:

$$a_h = \frac{2^{2n}}{(2h)!}(-1)^{n-1}(-1)^h[\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u}+$$

$$+2h\sum_{u=1}^{h}s(2h-1,2u-1)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u}+$$

$$+\sum_{j=2}^{2h}\binom{2h}{j}\sum_{u=0}^{2h-1}s(2h-1,u)\sum_{p=1}^{[(u+1)/2]}\binom{u}{2p-1}(j-1)^{u-2p+1}(2^{2n+2p}-1)\frac{B_{2n+2p}}{2n+2p}]. \qquad \text{(A.1.3.05)}$$

Ponendo, h = 0, nella (A.1.3.05), ritroviamo la formula (A.1.1.06):

cioè: $a_0 = 2^{2n}(2^{2n}-1)\frac{|B_{2n}|}{2n}$

Per h = n, ricaviamo:

$$a_n = -\frac{2^{2n}}{(2n)!}[\sum_{u=0}^{n}s(2n,2u)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u}+$$

$$+2n\sum_{u=1}^{n}s(2n-1,2u-1)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u}+$$

$$+\sum_{j=2}^{2n}\binom{2n}{j}\sum_{u=0}^{2n-1}s(2n-1,u)\sum_{p=1}^{[(u+1)/2]}\binom{u}{2p-1}(j-1)^{u-2p+1}(2^{2n+2p}-1)\frac{B_{2n+2p}}{2n+2p}]=\mathbf{(2n-1)!} \qquad \text{(A.1.3.06)}$$

Le relazioni (A.1.3.05) e (A.1.3.06) sono state verificate con un programma di matematica. E' straordinario verificare che un'espressione molto complessa, come quella di cui al 2° membro della (A.1.3.06), risulta uguale a **(2n-1)!**

A.1.4 –Osserviamo che ponendo, x = ¼, nella (A.1.0.01), troviamo:

$$\int_0^\infty \frac{u^{2n-1}Cosh(u/4)}{Sinh(u/2)}du = \pi^{2n}\sum_{h=0}^{n}a_h \ , \qquad \text{(A.1.4.01)}$$

Sviluppando i calcoli, abbiamo:

$$\int_0^\infty \frac{u^{2n-1}Cosh(u/4)}{Sinh(u/2)}du = \int_0^\infty u^{2n-1}(e^{u/4}+e^{-u/4})\sum_{k\geq 0}e^{-u/2}e^{-uk}du =$$

$$=\int_0^\infty\sum_{k\geq =}e^{-uk}u^{2n-1}(e^{-u/4}+e^{-3u/4})du = (2n-1)!\sum_{k\geq 0}[\frac{1}{(\frac{1}{4}+k)^{2n}}+\frac{1}{(\frac{3}{4}+k)^{2n}}]=$$

$$=(2n-1)!4^{2n}\sum_{k\geq 0}[\frac{1}{(1+4k)^{2n}}+\frac{1}{(3+4k)^{2n}}] = \pi^{2n}\sum_{h=0}^{n}a_h \ , \qquad \text{(A.1.4.02)}$$

da cui , tenendo presente la (A.1.2.01), ricaviamo:

$$\sum_{k\geq 0}[\frac{1}{(1+4k)^{2n}}+\frac{1}{(3+4k)^{2n}}] = \frac{\pi^{2n}}{2(2n)!}(2^{2n}-1)|B_{2n}| \qquad \text{(A.1.4.03)}$$

Inoltre, dalla (A.1.4.01) otteniamo:

$$\int_0^\infty \frac{u^{2n-1}Cosh(u/4)}{Sinh(u/2)}du = \int_0^\infty \frac{u^{2n-1}}{2Sinh(u/4)}du = \int_0^\infty u^{2n-1}\sum_{k=0} e^{-u/4}e^{-uk/2}du =$$

$$=\sum_{k=0}\frac{(2n-1)!}{(\frac{1}{4}+\frac{k}{2})^{2n}} = 4^{2n}(2n-1)!\sum_{k=0}\frac{1}{(1+2k)^{2n}} = \pi^{2n}\sum_{h=0}^{n} a_h \text{ , da cui:}$$

$$\sum_{k=0}\frac{1}{(1+2k)^{2n}} = \frac{\pi^{2n}}{2(2n)!}(2^{2n}-1)|B_{2n}| \qquad \text{(A.1.4.04)}$$

che è perfettamente identica alla (A.1.1.10).

Dal confronto tra la (A.1.4.04) e la (A.1.4.03), troviamo, com'era naturale, che:

$$\sum_{k=0}\frac{1}{(1+2k)^{2n}} = \sum_{k\geq 0}[\frac{1}{(1+4k)^{2n}}+\frac{1}{(3+4k)^{2n}}] \qquad \text{(A.1.4.05)}$$

A.1.5 – Considerando il caso generale fornito dalla (A.1.0.01), abbiamo:

$$\int_0^\infty \frac{u^{2n-1}Cosh(ux)}{Sinh(u/2)}du = \pi^{2n}\sum_{h=0}^{n} a_h[Tan(\pi x)]^{2h} = 2^{2n}\pi^{2n}(-1)^n\sum_{k\geq 1}(-1)^k k^{2n-1}e^{-2\pi i x k} \; ; \qquad \text{(A.1.5.01)}$$

Dall'integrale del 1° membro della (A.1.5.01), ricaviamo:

$$\int_0^\infty \frac{u^{2n-1}Cosh(ux)}{Sinh(u/2)}du = \int_0^\infty u^{2n-1}(e^{ux}+e^{-ux})\sum_{k\geq 0}e^{-u/2}e^{-uk}du =$$

$$=(2n-1)!\sum_{k\geq 0}[\frac{1}{(\frac{1}{2}-x+k)^{2n}}+\frac{1}{(\frac{1}{2}+x+k)^{2n}}] = \pi^{2n}\sum_{h=0}^{n} a_h[Tan(\pi x)]^{2h}, \qquad |x|<\frac{1}{2}; \qquad \text{(A.1.5.02)}$$

da cui: $$\sum_{k\geq 0}[\frac{1}{(\frac{1}{2}-x+k)^{2n}}+\frac{1}{(\frac{1}{2}+x+k)^{2n}}] = \frac{\pi^{2n}}{(2n-1)!}\sum_{h=0}^{n} a_h[Tan(\pi x)]^{2h} \; ; \qquad \text{(A.1.5.03)}$$

ricordiamo che a_n = **(2n-1)!** ;

dalla (1.5.02), per x = 0, ritroviamo la formula:

$$2^{2n+1}(2n-1)!\sum_{k\geq 0}\frac{1}{(1+2k)^{2n}} = \pi^{2n}a_0 = \pi^{2n}2^{2n}(2^{2n}-1)\frac{|B_{2n}|}{2n} \; ; \qquad \text{(A.1.5.04)}$$

cioè: $$\sum_{k\ge0}\frac{1}{(1+2k)^{2n}} = \frac{(2^{2n}-1)\pi^{2n}}{2(2n)!}|B_{2n}| \qquad \text{(A.1.5.05)}$$

identica alla (A.1.1.10).

Per x = ¼, ritroviamo la formula (A.1.4.03)).

Dall'ultimo membro della (A.1.5.01), ricaviamo:

$$2^{2n}\pi^{2n}(-1)^n\sum_{k\ge1}(-1)^k k^{2n-1}e^{-2\pi ixk} = 2^{2n}\pi^{2n}(-1)^n\sum_{k\ge1}(-1)^k k^{2n-1}\sum_{h\ge0}\frac{(-2i\pi xk)^h}{h!} =$$

$$=2^{2n}\pi^{2n}(-1)^n\sum_{h\ge0}\frac{(-2\pi ix)^h}{h!}\sum_{k\ge1}(-1)^k k^{2n-1+h} = 2^{2n}\pi^{2n}(-1)^n\sum_{h\ge0}\frac{(2\pi x)^{2h}(-1)^h}{(2h)!}\sum_{k\ge1}(-1)^k k^{2n-1+2h} ;$$

ricordiamo che $\sum_{k\ge1}(-1)^k k^{2n+2h} = 0$;

tenendo presente la (A.1.1.05), troviamo:

$$2^{2n}\pi^{2n}(-1)^n\sum_{k\ge1}(-1)^k k^{2n-1}e^{-2\pi ixk} = 2^{2n}\pi^{2n}(-1)^{n-1}\sum_{h\ge0}\frac{(2\pi x)^{2h}(-1)^h}{(2h)!}\frac{(2^{2n+2h}-1)B_{2n+2h}}{2n+2h}$$, e quindi:

$$(2n-1)!\sum_{k\ge0}\Big[\frac{1}{(\frac{1}{2}-x+k)^{2n}}+\frac{1}{(\frac{1}{2}+x+k)^{2n}}\Big] = 2^{2n}\pi^{2n}\sum_{h\ge0}\frac{(2\pi x)^{2h}}{(2h)!}\frac{(2^{2n+2h}-1)|B_{2n+2h}|}{2n+2h}, \qquad \text{(A.1.5.06)}$$

Dalla (A.1.5.06), ricaviamo:

$$\sum_{k\ge0}\Big[\frac{1}{(\frac{1}{2}-x+k)^{2n}}+\frac{1}{(\frac{1}{2}+x+k)^{2n}}\Big] = \frac{2^{2n}}{(2n-1)!}\pi^{2n}\sum_{h\ge0}\frac{(2\pi x)^{2h}}{(2h)!}\frac{(2^{2n+2h}-1)|B_{2n+2h}|}{2n+2h} =$$

$$=\frac{\pi^{2n}}{(2n-1)!}\sum_{h=0}^{n}a_h[Tan(\pi x)]^{2h} \qquad \text{(A.1.5.07)}$$

che, per x = 0, fornisce la (A.1.5.04), e per x = ¼, fornisce:

$$\sum_{k\ge0}\Big[\frac{1}{(1+4k)^{2n}}+\frac{1}{(3+4k)^{2n}}\Big] = \frac{\pi^{2n}}{2^{2n}(2n-1)!}\sum_{h\ge0}\frac{(\pi/2)^{2h}}{(2h)!}\frac{(2^{2n+2h}-1)|B_{2n+2h}|}{2n+2h} =$$

$$=\frac{\pi^{2n}}{2(2n)!}(2^{2n}-1)|B_{2n}| \qquad \text{(A.1.5.08)}$$

A.2.0- Derivata di ordine (2n) della funzione P(x) = Tan(x)

Derivando la (A.0.0.04), (2n) volte, rispetto ad x, troviamo:

$$\int_0^\infty \frac{u^{2n} Sinh(ux)}{Sinh(u/2)} du = \pi[Tan(\pi x)]^{(2n)} =$$

$$= \pi^{2n+1} \sum_{h=0}^{n} b_h [Tan(\pi x)]^{2h+1} = \frac{2\pi}{i} \sum_{k\geq 1} (-1)^k (-2\pi i)^{2n} k^{2n} e^{-2\pi i x k} \qquad \text{(A.2.0.01)}$$

Non è difficile verificare la relazione

$$\pi[Tan(\pi x)]^{(2n)} = \pi^{2n+1} \sum_{h=0}^{n} b_h [Tan(\pi x)]^{2h+1},$$

e la relazione $\qquad [Tan(x)]^{(2n)} = \sum_{h=0}^{n} b_h [Tan(x)]^{2h+1}$

A.2.1- Determinazione del coefficiente b_0 del polinomio rappresentato

dallo sviluppo della funzione $\quad [Tan(x)]^{(2n)} = \sum_{h=0}^{n} b_h [Tan(x)]^{2h+1}$

Operando sulla (A.2.0.01), abbiamo:

$$\int_0^\infty \frac{u^{2n} Sinh(ux)}{Sinh(u/2)} du = \pi[Tan(\pi x)]^{2n} =$$

$$= \pi^{2n+1} \sum_{h=0}^{n} b_h [Tan(\pi x)]^{2h+1} = \frac{(2\pi)^{2n+1}(-1)^n}{i} \sum_{k\geq 1} (-1)^k k^{2n} e^{-2\pi i x k} \qquad \text{(A.2.1.01)}$$

Ponendo, nella (A.2.1.01), $Tan(\pi x) = t$, da cui $\pi x = ArcTan(t)$, otteniamo:

$$\int_0^\infty \frac{u^{2n} Sinh[\frac{u}{\pi} ArcTan(t)]}{Sinh(u/2)} du = \pi^{2n+1} \sum_{h=0}^{n} b_h t^{2h+1} \qquad \text{(A.2.1.02)}$$

Derivando la precedente (A.2.1.02), rispetto a (t), e ponendo dopo, t = 0, troviamo:

$$\lim_{t\to 0} D_t \int_0^\infty \frac{u^{2n} Sinh[\frac{u}{\pi} ArcTan(t)]}{Sinh(u/2)} du = \frac{1}{\pi} \int_0^\infty \frac{u^{2n+1}}{Sinh(u/2)} du = \pi^{2n+1} b_0, \text{ da cui:}$$

$$\frac{2}{\pi} \int_0^\infty u^{2n+1} e^{-u/2} \sum_{k\geq 0} e^{-uk} du = \frac{2}{\pi} \sum_{k\geq 0} \frac{(2n+1)!}{(\frac{1}{2}+k)^{2n+2}} = \frac{2^{2n+3}(2n+1)!}{\pi} \sum_{k\geq 0} \frac{1}{(1+2k)^{2n+2}};$$

applicando la (A.1.1.07), troviamo: $\sum_{k\geq 0} \frac{1}{(1+2k)^{2n+2}} = [1-2^{-(2n+2)}]\zeta(2n+2)$, e quindi:

$$\pi^{2n+1} b_0 = \frac{2^{2n+3}(2n+1)!}{\pi} [1-2^{-(2n+2)}]\zeta(2n+2) = \frac{2(2n+1)!}{\pi} (2^{2n+2}-1)\zeta(2n+2);$$

ma: $\quad \zeta(2n+2) = \frac{2^{2n+1}\pi^{2n+2}}{(2n+2)!} |B_{2n+2}|$, e pertanto:

$$b_0 = \frac{2^{2n+1}}{n+1} (2^{2n+2}-1) |B_{2n+2}| \qquad \text{(A.2.1.03)}$$

La relazione (A.2.1.03) è stata verificata con un programma di matematica.

Riportiamo i primi valori di b_0 per n variabile da 1 a 10.
n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
b_0 = 2, 16, 272, 7936, 353792, 22368256, 1903757312, 209865342476, 29088885112832, 4951498053124096

A.**2.2 -Determinazione dell'espressione** della Somma $\sum_{h=0}^{n} b_h$, in funzione di n.

Utilizzando la relazione (A.2.1.01), e ponendo, x = ¼, troviamo:

$$\int_0^\infty \frac{u^{2n} Sinh(u/4)}{Sinh(u/2)} du = \pi^{2n+1} \sum_{h=0}^{n} b_h = \frac{(2\pi)^{2n+1}(-1)^n}{i} \sum_{k\geq 1} (-1)^k k^{2n} (-i)^k , \qquad \text{(A.2.2.01)}$$

da cui:

$$\sum_{h=0}^{n} b_h = \frac{2^{2n+1}(-1)^n}{i} \sum_{k\geq 1} (i)^k k^{2n} \text{ , ma } \sum_{k\geq 1} (i)^k k^{2n} = \sum_{k\geq 1} (-1)^k (2k)^{2n} + \sum_{k\geq 1} (i)^{2k-1} (2k-1)^{2n} \text{ ;}$$

quindi: $\sum_{h=0}^{n} b_h = \frac{2^{2n+1}(-1)^n}{i} [\sum_{k\geq 1} (-1)^k (2k)^{2n} + \sum_{k\geq 1} (i)^{2k-1} (2k-1)^{2n}]$.

Dalla precedente, uguagliando le parti reali, ricaviamo:

$$\sum_{h=0}^{n} b_h = \frac{2^{2n+1}(-1)^n}{i} \sum_{k\geq 1} (i)^{2k-1} (2k-1)^{2n} = 2^{2n+1} (-1)^{n-1} \sum_{k\geq 1} (-1)^k (2k-1)^{2n} \text{; inoltre,}$$

$$\sum_{k\geq 1} (-1)^k (2k-1)^{2n} = \sum_{k\geq 1} (-1)^k (1-2k)^{2n} = \sum_{k\geq 1} (-1)^k \sum_{h\geq 0} \binom{2n}{h} (-2k)^h =$$

$$= \sum_{k\geq 1} (-1)^k + \sum_{h\geq 1} \binom{2n}{h} (-2)^h \sum_{k\geq 1} (-1)^k k^h \text{ ;}$$

poiché $\sum_{k\geq 1} (-1)^k = -\frac{1}{2}$, (Ved. [4], pag.2) (A.2.2.02)

e ricordando che: $\zeta(-2h) = 0$, (h = 1,2,3…,n), otteniamo:

$$\sum_{k\geq 1} (-1)^k (2k-1)^{2n} = -\frac{1}{2} + \sum_{h\geq 1} \binom{2n}{2h-1} (-2)^{2h-1} \sum_{k\geq 1} (-1)^k k^{2h-1}$$

Ricordando la (A.1.1.05), abbiamo :

$$\sum_{k\geq 1} (-1)^k (2k-1)^{2n} = -\frac{1}{2} - \sum_{h\geq 1} \binom{2n}{2h-1} (-2)^{2h-1} (2^{2h}-1) \frac{B_{2h}}{2h} \text{ ;}$$

osserviamo che: $\binom{2n}{2h-1} \frac{1}{2h} = \frac{1}{2n+1} \binom{2n+1}{2h}$; pertanto:

$$\sum_{k\geq 1} (-1)^k (2k-1)^{2n} = -\frac{1}{2} + \frac{1}{2(2n+1)} \sum_{h\geq 1} \binom{2n+1}{2h} (-2)^{2h} (2^{2h}-1) B_{2h} , \qquad \text{(A.2.2.03)}$$

e quindi:

$$\sum_{h=0}^{n} b_h = 2^{2n} (-1)^{n-1} [\frac{1}{2n+1} \sum_{h=1}^{n} \binom{2n+1}{2h} 2^{2h} (2^{2h}-1) B_{2h} - 1] \qquad \text{(A.2.2.04)}$$

La relazione (A.2.2.04) è stata verificata con un programma di matematica.

Riportiamo i primi valori di $\sum_{h=0}^{n} b_h$ per n variabile da 1 a 10.

n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

$\sum_{h=0}^{n} b_h$ = 4, 80, 3904, 354560, 51733504, 11070525440, 3266330312704,

1270842139934720, 630424777638805504, 388362339077351014400

A.2.3 - Determinazione delle espressioni che definiscono, in funzione di n, i coefficienti $b_1, b_2, b_3, ..., b_n$ dei termini del polinomio $\sum_{h=0}^{n} b_h [Tan(x)]^{2h+1}$

Ponendo, nella (A.2.1.01), $Tan(\pi x) = t$, da cui $\pi x = ArcTan(t) = \frac{1}{2i} \ln \frac{1+it}{1-it}$, otteniamo:

$$\pi^{2n+1} \sum_{h=0}^{n} b_h t^{2h+1} = \frac{(2\pi)^{2n+1}(-1)^n}{i} \sum_{k\geq 1} (-1)^k k^{2n} e^{-k \ln\frac{1+it}{1-it}} =$$

$$= \frac{(2\pi)^{2n+1}(-1)^n}{i} \sum_{k\geq 1} (-1)^k k^{2n} (\frac{1-it}{1+it})^k ; \qquad \text{(A.2.3.01)}$$

derivando la (A.2.3.01), (2h+1) volte, rispetto a t, e ponendo dopo, t = 0, abbiamo:

$$\pi^{2n+1} b_h (2h+1)! = [\frac{(2\pi)^{2n+1}(-1)^n}{i} \sum_{k\geq 1} (-1)^k k^{2n} (\frac{1-it}{1+it})^k]_{t=0}^{(2h+1)} ;$$

ora, $[(1-it)^k (1+it)^{-k}]_{t=0}^{(2h+1)} = \lim_{t \to 0} \sum_{j=0}^{2h+1} \binom{2h+1}{j} [(1-it)^k]^{(2h+1-j)} [(1+it)^{-k}]^{(j)} =$

$$= \sum_{j=0}^{2h+1} \binom{2h+1}{j} \frac{\Gamma(k+1)(-i)^{2h+1-j}}{\Gamma(k+1-2h-1+j)} \frac{\Gamma(k+j)(-i)^j}{\Gamma(k)} =$$

$$= \sum_{j=0}^{2h+1} \binom{2h+1}{j} (-1)^h (-i) k \frac{\Gamma(k+j)}{\Gamma(k+j-2h)} =$$

$$= \sum_{j=0}^{2h+1} \binom{2h+1}{j} (-1)^h (-i) k (k+j-1)(k+j-2)...(k+j-2h) =$$

$$= \sum_{j=0}^{2h+1} \binom{2h+1}{j} (-1)^h (-i) k \sum_{u=0}^{2h} s(2h,u)(k+j-1)^u =$$

$$= \sum_{j=0}^{2h+1} \binom{2h+1}{j} (-1)^h (-i) k \sum_{u=0}^{2h} s(2h,u) \sum_{p=0}^{u} \binom{u}{p} k^p (j-1)^{u-p} ; \text{ pertanto:}$$

$$b_h (2h+1)! =$$

$$= \frac{2^{2n+1}(-1)^n}{i} \sum_{k\geq 1} (-1)^k k^{2n} \sum_{j=0}^{2h+1} \binom{2h+1}{j} (-1)^h (-i) k \sum_{u=0}^{2h} s(2h,u) \sum_{p=0}^{u} \binom{u}{p} k^p (j-1)^{u-p} =$$

$$= 2^{2n+1} (-1)^{n-1} (-1)^h \sum_{j=0}^{2h+1} \binom{2h+1}{j} \sum_{u=0}^{2h} s(2h,u) \sum_{p=0}^{u} \binom{u}{p} (j-1)^{u-p} \sum_{k\geq 1} (-1)^k k^{2n+p+1} =$$

$$=2^{2n+1}(-1)^{n-1}(-1)^{h}\sum_{j=0}^{2h+1}\binom{2h+1}{j}\sum_{u=0}^{2h}s(2h,u)\sum_{p=0}^{[u/2]}\binom{u}{2p}(j-1)^{u-2p}\sum_{k\geq 1}(-1)^{k}k^{2n+2p+1}=$$

$$=2^{2n+1}(-1)^{n+h}\sum_{j=0}^{2h+1}\binom{2h+1}{j}\sum_{u=0}^{2h}s(2h,u)\sum_{p=0}^{[u/2]}\binom{u}{2p}(j-1)^{u-2p}(2^{2n+2p+2}-1)\frac{B_{2n+2p+2}}{2n+2p+2}. \qquad \text{(A.2.3.02)}$$

Anche qui, osserviamo che nello sviluppo della relazione precedente (A.2.3.02), al variare di j, u, p, si presentano valori indeterminati (0^0).
Per eliminare detti inconvenienti calcoliamo l'espressione precedente per j = 0, per j = 1, e per j > 1. Così operando, troviamo:

1) per j = 0, $\sum_{j=0}^{2h+1}\binom{2h+1}{j}(-1)^{h}(-i)k(k+j-1)(k+j-2)...(k+j-2h)=$

$$=(-1)^{h}(-i)k(k-1)(k-2)...(k-2h)=(-1)^{h}(-i)\sum_{u=0}^{2h+1}s(2h+1,u)k^{u}$$

$$[b_h(2h+1)!]_{j=0}=[\frac{2^{2n+1}(-1)^{n}}{i}\sum_{k\geq 1}(-1)^{k}k^{2n}\ (-1)^{h}(-i)\sum_{u=0}^{2h+1}s(2h+1,u)k^{u}=$$

$$=2^{2n+1}(-1)^{n-1}(-1)^{h}\sum_{u=0}^{2h+1}s(2h+1,u)\sum_{k\geq 1}(-1)^{k}k^{2n+u}=$$

$$=2^{2n+1}(-1)^{n-1}(-1)^{h}\sum_{u=0}^{h}s(2h+1,2u+1)\sum_{k\geq 1}(-1)^{k}k^{2n+2u+1}=$$

$$=2^{2n+1}(-1)^{n}(-1)^{h}\sum_{u=0}^{h}s(2h+1,2u+1)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}$$

2) per j = 1, $\sum_{j=0}^{2h+1}\binom{2h+1}{j}(-1)^{h}(-i)k(k+j-1)(k+j-2)...(k+j-2h)=$

$$=(2h+1)(-1)^{h}(-i)k(k)(k-1)...(k+1-2h)=(2h+1)(-1)^{h}(-i)k\sum_{u=0}^{2h}s(2h,u)k^{u}$$

$$[b_h(2h+1)!]_{j=1}=\frac{2^{2n+1}(-1)^{n}}{i}\sum_{k\geq 1}(-1)^{k}k^{2n}\ (2h+1)(-1)^{h}(-i)k\sum_{u=0}^{2h}s(2h,u)k^{u}=$$

$$=2^{2n+1}(-1)^{n-1}(2h+1)(-1)^{h}\sum_{u=0}^{2h}s(2h,u)\sum_{k\geq 1}(-1)^{k}k^{2n+1+u}=$$

$$=2^{2n+1}(-1)^{n-1}(2h+1)(-1)^{h}\sum_{u=0}^{h}s(2h,2u)\sum_{k\geq 1}(-1)^{k}k^{2n+1+2u}=$$

$$=2^{2n+1}(-1)^{n}(2h+1)(-1)^{h}\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2};$$

Pertanto, ricaviamo:

$$b_h(2h+1)!=2^{2n+1}(-1)^{n}(-1)^{h}\sum_{u=0}^{h}s(2h+1,2u+1)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}+$$

$$+2^{2n+1}(-1)^n(2h+1)(-1)^h\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}+$$

$$+2^{2n+1}(-1)^{n+h}\sum_{j=2}^{2h+1}\binom{2h+1}{j}\sum_{u=0}^{2h}s(2h,u)\sum_{p=0}^{[u/2]}\binom{u}{2p}(j-1)^{u-2p}(2^{2n+2p+2}-1)\frac{B_{2n+2p+2}}{2n+2p+2},$$

da cui: $b_h=\dfrac{2^{2n+1}}{(2h+1)!}(-1)^n(-1)^h[\sum_{u=0}^{h}s(2h+1,2u+1)(2^{2n+2u+2}-1)\dfrac{B_{2n+2u+2}}{2n+2u+2}+$

$$+(2h+1)\sum_{u=0}^{h}s(2h,2u)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}+$$

$$+\sum_{j=2}^{2h+1}\binom{2h+1}{j}\sum_{u=0}^{2h}s(2h,u)\sum_{p=0}^{[u/2]}\binom{u}{2p}(j-1)^{u-2p}(2^{2n+2p+2}-1)\frac{B_{2n+2p+2}}{2n+2p+2}] \qquad \text{(A.2.3.03)}$$

Per h = 0, ritroviamo la formula(A.2.1.03): $b_0 = \dfrac{2^{2n+1}}{n+1}(2^{2n+2}-1)|B_{2n+2}|$

Per h = n, ricaviamo:

$$b_n=\frac{2^{2n+1}}{(2n+1)!}[\sum_{u=0}^{n}s(2n+1,2u+1)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}+ \qquad +$$

$$+(2n+1)\sum_{u=0}^{n}s(2n,2u)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}$$

$$+\sum_{j=2}^{2n+1}\binom{2n+1}{j}\sum_{u=0}^{2n}s(2n,u)\sum_{p=0}^{[u/2]}\binom{u}{2p}(j-1)^{u-2p}(2^{2n+2p+2}-1)\frac{B_{2n+2p+2}}{2n+2p+2}] = \mathbf{(2n)!} \qquad \text{(A.2.3.04)}$$

Le relazioni (A.2.3.03) e (a.2.3.04) sono state verificate con un programma di matematica.
E' straordinario verificare che un'espressione molto complessa, come quella di cui al 2° membro della (A.2.3.04), risulta uguale a **(2n)!**

A.2.4 – Consideriamo la relazione (A.2.1.01),

$$\int_0^\infty \frac{u^{2n}Sinh(ux)}{Sinh(u/2)}du = \pi[Tan(\pi x)]^{2n} =$$

$$=\pi^{2n+1}\sum_{h=0}^{n}b_h[Tan(\pi x)]^{2h+1} = \frac{(2\pi)^{2n+1}(-1)^n}{i}\sum_{k\geq 0}(-1)^k(1+k)^{2n}e^{-2\pi i x(1+k)};$$

ponendo, in detta relazione, x = 0, otteniamo:

$$\sum_{k\geq 0}(-1)^k(1+k)^{2n} = 0; \qquad \text{(A.2.4.01)}$$

$$\sum_{k\geq 0}(-1)^k(1+k)^{2n} = 1+\sum_{k\geq 1}(-1)^k\sum_{h\geq o}\binom{2n}{h}k^h = 1+\sum_{k\geq 1}(-1)^k+\sum_{h\geq 1}\binom{2n}{h}\sum_{k\geq 1}(-1)^k k^h = 0;$$

per la (A.2.2.02), $\sum_{k\geq 1}(-1)^k = -\frac{1}{2}$; e $\zeta(-2h)=0$, (h = 1,2,3,…); pertanto:

$$\sum_{k\geq 0}(-1)^k(1+k)^{2n} = \frac{1}{2}+\sum_{h\geq 1}\binom{2n}{2h-1}\sum_{k\geq 1}(-1)^k k^{2h-1};$$

per la (A.1.1.05), otteniamo:

$$\sum_{k\geq 0}(-1)^k(1+k)^{2n} = \frac{1}{2}+\sum_{h\geq 1}\binom{2n}{2h-1}(2^{2h}-1)(-\frac{B_{2h}}{2h})=0$$

ricordando che: $\binom{2n}{2h-1}\frac{1}{2h}=\frac{1}{2n+1}\binom{2n+1}{2h}$, troviamo:

$\frac{1}{2}-\frac{1}{2n+1}\sum_{h\geq 1}\binom{2n+1}{2h}(2^{2h}-1)B_{2h}=0$, da cui:

$$\frac{2}{2n+1}\sum_{h=1}^{n}\binom{2n+1}{2h}(2^{2h}-1)B_{2h}=1 \qquad \text{(A.2.4.02)}$$

La formula (A.2.4.02) è stata verificata con un programma di matematica.

A.2.5 - Ponendo nella relazione (A.2.1.01), x = ¼, troviamo:

$$\int_0^\infty \frac{u^{2n}Sinh(u/4)}{Sinh(u/2)}du = \pi[Tan(\pi x)]_{x=1/4}^{(2n)} = \frac{-(2\pi)^{2n+1}(-1)^n}{i}\sum_{k\geq 0}(-1)^k(1+k)^{2n}e^{-2\pi i(1+k)/4};$$

$$\int_0^\infty \frac{u^{2n}Sinh(u/4)}{Sinh(u/2)}du = \int_0^\infty \frac{u^{2n}}{2Cosh(u/4)}du = \int_0^\infty u^{2n}e^{-u/4}\sum_{k\geq 0}(-1)^k e^{-uk/2}du =$$

$$=\sum_{k\geq 0}(-1)^k\frac{(2n)!}{(\frac{1}{4}+\frac{k}{2})^{2n+1}} = 4^{2n+1}(2n)!\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}};$$

$$\frac{-(2\pi)^{2n+1}(-1)^n}{i}\sum_{k\geq 0}(-1)^k(1+k)^{2n}e^{-2\pi i(1+k)/4} = 2^{2n+1}\pi^{2n+1}(-1)^n\sum_{k\geq 0}(i)^k(1+k)^{2n} =$$

$$=2^{2n+1}\pi^{2n+1}(-1)^n[\sum_{k\geq 0}(-1)^k(1+2k)^{2n}+\sum_{k\geq 1}(i)^{2k-1}(2k)^{2n}] = 4^{2n+1}(2n)!\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}};$$

uguagliando le parti reali, troviamo:

$$4^{2n+1}(2n)!\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}} = \pi^{2n+1}\sum_{h=0}^{n}b_h = 2^{2n+1}\pi^{2n+1}(-1)^n\sum_{k\geq 0}(-1)^k(1+2k)^{2n};$$

$$\sum_{k\geq 0}(-1)^k(1+2k)^{2n} = 1+\sum_{k\geq 1}(-1)^k\sum_{h\geq 0}\binom{2n}{h}(2k)^h = 1+\sum_{k\geq 1}(-1)^k+\sum_{h\geq 1}\binom{2n}{h}2^h\sum_{k\geq 1}(-1)^k k^h;$$

per la (A.2.2.02), e ricordando che $\zeta(-2h)=0$, (h = 1,2,3,…), otteniamo:

$$\sum_{k\geq 0}(-1)^k(1+2k)^{2n} = \frac{1}{2}+\sum_{h\geq 1}\binom{2n}{2h-1}2^{2h-1}\sum_{k\geq 1}(-1)^k k^{2h-1} =$$

$$=\frac{1}{2}-\frac{1}{2(2n+1)}\sum_{h\geq 1}\binom{2n+1}{2h}2^{2h}(2^{2h}-1)B_{2h}\,; \qquad \text{(A.2.5.01)}$$

quindi: $4^{2n+1}(2n)!\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}}=\pi^{2n+1}\sum_{h=0}^{n}b_h=$

$=2^{2n+1}\pi^{2n+1}(-1)^{n-1}[-\frac{1}{2}+\frac{1}{2(2n+1)}\sum_{h\geq 1}\binom{2n+1}{2h}2^{2h}(2^{2h}-1)B_{2h}]$; cioè:

$$\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}}=\frac{\pi^{2n+1}(-1)^{n-1}}{2^{2n+2}(2n+1)!}[\sum_{h=1}^{n}\binom{2n+1}{2h}2^{2h}(2^{2h}-1)B_{2h}-(2n+1)] \qquad \text{(A.2.5.02)}$$

Osserviamo che:

$$\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}}=\sum_{k\geq 0}[\frac{1}{(1+4k)^{2n+1}}-\frac{1}{(3+4k)^{2n+1}}]$$

La formula (A.2.5.02) è stata verificata con un programma di matematica.

A.2.6. – Consideriamo il caso generale fornito dalla relazione (A.2.1.01)

$$\int_0^{\infty}\frac{u^{2n}Sinh(ux)}{Sinh(u/2)}du=\pi[Tan(\pi x)]^{(2n)}=\frac{-(2\pi)^{2n+1}(-1)^n}{i}\sum_{k\geq 0}(-1)^k(1+k)^{2n}e^{-2\pi i x(1+k)}$$

Sviluppando, abbiamo:

$$\int_0^{\infty}\frac{u^{2n}Sinh(ux)}{Sinh(u/2)}du=\int_0^{\infty}u^{2n}(e^{ux}-e^{-ux})e^{-u/2}\sum_{k\geq 0}e^{-ux}du=$$

$$=(2n)!\sum_{k\geq 0}[\frac{1}{(\frac{1}{2}-x+k)^{2n+1}}-\frac{1}{(\frac{1}{2}+x+k)^{2n+1}}]=\pi^{2n+1}\sum_{h=0}^{n}b_h[Tan(x)]^{2h+1}\,; \qquad \text{(A.2.6.01)}$$

Inoltre:

$$\frac{(2\pi)^{2n+1}(-1)^n}{i}\sum_{k\geq 1}(-1)^k k^{2n}e^{-2\pi i x k}=$$

$$=\frac{(2\pi)^{2n+1}(-1)^n}{i}\sum_{k\geq 1}(-1)^k k^{2n}\sum_{h\geq 0}\frac{(-2\pi i x k)^h}{h!}\,; \qquad \text{(A.2.6.02)}$$

per x reale, l'ultimo membro della (A.2.6.02) è reale solamente per h dispari; quindi:

$$\pi^{2n+1}\sum_{h=0}^{n}b_h[Tan(x)]^{2h+1}=\frac{(2\pi)^{2n+1}(-1)^n}{i}\sum_{k\geq 1}(-1)^k k^{2n}\sum_{h\geq 1}\frac{(-2\pi i x)^{2h-1}k^{2h-1}}{(2h-1)!}=$$

$$=(2\pi)^{2n+1}(-1)^n\sum_{h\geq 1}\frac{(2\pi x)^{2h-1}(-1)^h}{(2h-1)!}\sum_{k\geq 0}(-1)^k(k)^{2n+2h-1}=$$

$$=(2\pi)^{2n+1}(-1)^{n-1}\sum_{h\geq 1}\frac{(2\pi x)^{2h-1}(-1)^h}{(2h-1)!}\frac{(2^{2n+2h}-1)B_{2n+2h}}{2n+2h}$$

Pertanto: $(2n)!\sum_{k\geq 0}[\frac{1}{(\frac{1}{2}-x+k)^{2n+1}}-\frac{1}{(\frac{1}{2}+x+k)^{2n+1}}] = \pi^{2n+1}\sum_{h=0}^{n} b_h [Tan(x)]^{2h+1} =$

$$= (2\pi)^{2n+1}(-1)^{n-1}\sum_{h\geq 1}\frac{(2\pi x)^{2h-1}(-1)^h}{(2h-1)!}\frac{(2^{2n+2h}-1)B_{2n+2h}}{2n+2h} \qquad \text{(A.2.6.03)}$$

Per, x = ¼, troviamo: $(2n)!4^{2n+1}\sum_{k\geq 0}[\frac{1}{(1+4k)^{2n+1}}-\frac{1}{(3+4k)^{2n+1}}] = \pi^{2n+1}\sum_{h=0}^{n} b_h =$

$$=(2\pi)^{2n+1}\sum_{h\geq 1}\frac{(\pi/2)^{2h-1}}{(2h-1)!}\frac{(2^{2n+2h}-1)|B_{2n+2h}|}{2n+2h} \qquad \text{(A.2.6.04)}$$

da cui: $\sum_{k\geq 0}[\frac{1}{(1+4k)^{2n+1}}-\frac{1}{(3+4k)^{2n+1}}] = \sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}} = \frac{\pi^{2n+1}}{(2n)!4^{2n+1}}\sum_{h=0}^{n} b_h =$

$$=\frac{(2\pi)^{2n+1}}{(2n)!4^{2n+1}}\sum_{h\geq 1}\frac{(\pi/2)^{2h-1}}{(2h-1)!}\frac{(2^{2n+2h}-1)|B_{2n+2h}|}{2n+2h} \qquad \text{(A.2.6.05)}$$

Dalla (A.2.6.04), otteniamo:

$$\frac{2^{2n+1}}{(2n)!4^{2n+1}}\sum_{h\geq 1}\frac{(\pi/2)^{2h-1}}{(2h-1)!}\frac{(2^{2n+2h}-1)|B_{2n+2h}|}{2n+2h} = \sum_{h=0}^{n} b_h \qquad \text{(A.2.6.06)}$$

B.0.0 – Caso B - C(x) = Sec(x)

Utilizzando la seguente relazione dei complementi:

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{Sin(\pi z)}, \qquad 0 < z < 1$$

e ponendo, $z=\frac{1}{2}-x$, (x, reale), otteniamo:

$$\Gamma(x+\frac{1}{2})\Gamma(\frac{1}{2}-x)=\frac{\pi}{Cos(\pi x)}, \qquad |x|<\frac{1}{2} \qquad \text{(B.0.0.01)}$$

Ora, $\Gamma(x+\frac{1}{2})\Gamma(\frac{1}{2}-x)=\int_0^1 t^{x-1+1/2}(1-t)^{-x-1+1/2}dt=\int_0^\infty t^{x-1+1/2}\frac{dt}{1+t}= (t=e^u)=$

$=\int_{-\infty}^\infty e^{u(x-1+1/2)}\frac{e^u du}{1+e^u}= \int_0^\infty e^{u(x-1+1/2)}\frac{e^u du}{1+e^u}+\int_0^\infty e^{-u(x-1+1/2)}\frac{e^{-u} du}{1+e^{-u}}=$

$\int_0^\infty e^{u(x+1/2)}\frac{du}{1+e^u}+\int_0^\infty e^{-u(x+1/2)}\frac{du}{1+e^{-u}}=$

$=\int_0^\infty e^{ux}\frac{du}{e^{u/2}+e^{-u/2}}+\int_0^\infty e^{-ux}\frac{du}{e^{u/2}+e^{-u/2}}=$

$$=\int_0^\infty\frac{Cosh(ux)}{Cosh(u/2)}du=\pi Sec(\pi x)=\frac{\pi}{Cos(\pi x)}, \qquad |x|<\frac{1}{2} \qquad \text{(B.0.0.02)}$$

B.1.1- Derivando, (2n-1) volte, rispetto ad x, la relazione (B.0.0.02), otteniamo:

$$\int_0^\infty \frac{u^{2n-1} Sinh(ux)}{Cosh(u/2)} du = [\frac{\pi}{Cos(\pi x)}]^{(2n-1)}, \qquad |x| < \frac{1}{2}, \tag{B.1.1.01}$$

Dalla (B.1.1.01), passando al limite, per x $\rightarrow$ 0, abbiamo:

$$\lim_{x \to 0} \int_0^\infty \frac{u^{2n-1} Sinh(ux)}{Cosh(u/2)} du = [\frac{\pi}{Cos(\pi x)}]_{x=0}^{(2n-1)} = 0;$$ cioè:

$$[\frac{\pi}{Cos(\pi x)}]_{x=0}^{(2n-1)} = 2\pi[\sum_{k \geq 0} (-)^k e^{-i\pi x(1+2k)}]_{x=0}^{(2n-1)} = 2\pi \sum_{k \geq 0} (-)^k (-i\pi)^{2n-1} (1+2k)^{2n-1} = 0,$$

da cui: $\sum_{k \geq 0} (-1)^k (1+2k)^{2n-1} = 0$; (B.1.1.02)

ora, $\sum_{k \geq 0} (-1)^k (1+2k)^{2n-1} = 1 + \sum_{k \geq 1} (-1)^k (1+2k)^{2n-1} = 1 + \sum_{k \geq 1} (-1)^k \sum_{h=0}^{2n-1} \binom{2n-1}{h} (2k)^h =$

$$=1+\sum_{k \geq 1} (-1)^k + \sum_{k \geq 1} (-1)^k \sum_{h=1}^{2n-1} \binom{2n-1}{h} (2k)^h = 1 - \frac{1}{2} + \sum_{h=1}^{2n-1} \binom{2n-1}{h} 2^h \sum_{k \geq 1} (-1)^k k^h =$$

$$=\frac{1}{2} + \sum_{h=1}^{n} \binom{2n-1}{2h-1} 2^{2h-1} \sum_{k \geq 1} (-1)^k k^{2h-1} = \frac{1}{2} - \frac{1}{2} \sum_{h=1}^{n} \binom{2n-1}{2h-1} 2^{2h} (2^{2h}-1) \frac{B_{2h}}{2h} = 0,$$ da cui:

$$\frac{1}{2n} \sum_{h=1}^{n} \binom{2n}{2h} 2^{2h} (2^{2h}-1) B_{2h} = 1 \tag{B.1.1.03}$$

Nel procedimento per ricavare la (B.1.1.03) abbiamo utilizzato le seguenti note formule:

(A.1.1.05), (A.2.2.02), $\zeta(1-2h) = -\frac{B_{2h}}{2h}$, $\zeta(-2h) = 0$, $\binom{2n-1}{2h-1} \frac{1}{2h} = \frac{1}{2n} \binom{2n}{2h}$

La relazione (B.1.1.03) è stata verificata con un programma di matematica.

B.1.2– Dalla (B.1.1.01), passando al limite, per x $\rightarrow$ 1/4, abbiamo:

$$\lim_{x \to 1/4} \int_0^\infty \frac{u^{2n-1} Sinh(ux)}{Cosh(u/2)} du = [\frac{\pi}{Cos(\pi x)}]_{x=1/4}^{(2n-1)} = 2\pi[\sum_{k \geq 0} (-1)^k e^{-i\pi x(1+2k)}]_{x=1/4}^{(2n-1)};$$

ora, $\int_0^\infty \frac{u^{2n-1} Sinh(u/4)}{Cosh(u/2)} du = \int_0^\infty u^{2n-1} (e^{u/4} - e^{-u/4}) e^{-u/2} \sum_{k \geq 0} (-1)^k e^{-uk} du =$

$$\int_0^\infty u^{2n-1}\sum_{k\geq 0}(-1)^k(e^{-u/4}-e^{-3u/4})e^{-uk}du = (2n-1)!\sum_{k\geq 0}(-1)^k[\frac{1}{(\frac{1}{4}+k)^{2n}}-\frac{1}{(\frac{3}{4}+k)^{2n}}]=$$

$=(2n-1)!4^{2n}\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n}}-\frac{1}{(3+4k)^{2n}}]$, mentre:

$$2\pi[\sum_{k\geq 0}(-1)^k e^{-i\pi x(1+2k)}]^{(2n-1)}_{x=1/4)} = 2\pi\sum_{k\geq 0}(-1)^k(-i\pi)^{2n-1}(1+2k)^{2n-1}e^{-i\pi/4}(-i)^k =$$

$=2\pi^{2n}\frac{(-1)^n}{-i}e^{-i\pi/4}\sum_{k\geq 0}(i)^k(1+2k)^{2n-1}$;

$$\sum_{k\geq 0}(i)^k(1+2k)^{2n-1} = \sum_{k\geq 0}(-1)^k(1+4k)^{2n-1}+\sum_{k\geq 1}(i)^{2k-1}(4k-1)^{2n-1} =$$

$=\sum_{k\geq 0}(-1)^k(1+4k)^{2n-1}+i\sum_{k\geq 1}(-1)^k(1-4k)^{2n-1}$; quindi:

$$2\pi^{2n}\frac{(-1)^n}{-i}e^{-i\pi/4}\sum_{k\geq 0}(i)^k(1+2k)^{2n-1} = \pi^{2n}(-1)^n\sqrt{2}(1+i)\sum_{k\geq 0}(i)^k(1+2k)^{2n-1} =$$

$$=\pi^{2n}(-1)^n\sqrt{2}(1+i)\ [\sum_{k\geq 0}(-1)^k(1+4k)^{2n-1}+i\sum_{k\geq 1}(-1)^k(1-4k)^{2n-1}\ ]$$

Prendendo la parte reale della precedente espressione, abbiamo:

$$\pi^{2n}(-1)^n\sqrt{2}\ [\sum_{k\geq 0}(-1)^k(1+4k)^{2n-1}-\sum_{k\geq 1}(-1)^k(1-4k)^{2n-1}\ ]=$$

$$=\pi^{2n}(-1)^n\sqrt{2}\ [1+\sum_{k\geq 1}(-1)^k\sum_{h=0}^{2n-1}\binom{2n-1}{h}(4k)^h-\sum_{k\geq 1}(-1)^k\sum_{h=0}^{2n-1}\binom{2n-1}{h}(-4k)^h\ ]=$$

$$=\pi^{2n}(-1)^n\sqrt{2}\ [1+\sum_{k\geq 1}(-1)^k\sum_{h=1}^{n}\binom{2n-1}{2h-1}(4k)^{2h-1}-\sum_{k\geq 1}(-1)^k\sum_{h=1}^{n}\binom{2n-1}{2h-1}(-4k)^{2h-1}\ ]=$$

$$=\pi^{2n}(-1)^n\sqrt{2}[1+\frac{1}{2}\sum_{h=1}^{n}\binom{2n-1}{2h-1}4^{2h}\sum_{k\geq 1}(-1)^k k^{2h-1}]=$$

$$=\pi^{2n}(-1)^n\sqrt{2}[1+\frac{1}{2}\sum_{h=1}^{n}\binom{2n-1}{2h-1}4^{2h}(2^{2h}-1)(-\frac{B_{2h}}{2h})]=$$

$=\pi^{2n}(-1)^{n-1}\sqrt{2}[\frac{1}{4n}\sum_{h=1}^{n}\binom{2n}{2h}4^{2h}(2^{2h}-1)B_{2h}-1]$; pertanto:

$$(2n-1)!4^{2n}\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n}}-\frac{1}{(3+4k)^{2n}}]=$$

$$=\pi^{2n}(-1)^{n-1}\sqrt{2}[\frac{1}{4n}\sum_{h=1}^{n}\binom{2n}{2h}4^{2h}(2^{2h}-1)B_{2h}-1]\text{, da cui:}$$

$$\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n}}-\frac{1}{(3+4k)^{2n}}]=$$

$$=\pi^{2n}(-1)^{n-1}\sqrt{2}\,\frac{4^{-2n}}{2(2n)!}[\sum_{h=1}^{n}\binom{2n}{2h}4^{2h}(2^{2h}-1)B_{2h}-4n]. \qquad \text{(B.1.2.01)}$$

La relazione (B.1.2.01) è stata verificata con un programma di matematica.

B.1.3 – Riprendiamo la (B.1.1.01)

$$\int_0^{\infty}\frac{u^{2n-1}Sinh(ux)}{Cosh(u/2)}du = [\frac{\pi}{Cos(\pi x)}]^{(2n-1)}, \qquad |x|<\frac{1}{2} \qquad \text{(B.1.3.01)}$$

Operando, abbiamo:

$$\int_0^{\infty}\frac{u^{2n-1}Sinh(ux)}{Cosh(u/2)}du = \int_0^{\infty}u^{2n-1}(e^{ux}-e^{-ux})\sum_{k\geq 0}e^{-u/2}(-1)^k e^{-uk}du =$$

$$=(2n-1)!\sum_{k\geq 0}(-1)^k[\frac{1}{(\frac{1}{2}-x+k)^{2n}}-\frac{1}{(\frac{1}{2}+x+k)^{2n}}]; \qquad \text{(B.1.3.02)}$$

$$[\frac{\pi}{Cos(\pi x)}]^{(2n-1)} = [\frac{\pi}{Cos(\pi x)}]^{(2n-2+1)} = \pi[\frac{\pi Sin(\pi x)}{[Cos(\pi x)]^2}]^{(2n-2)} =$$

$$=\pi[Sin(\pi x)DTan(\pi x)]^{(2n-2)}=\pi\sum_{k=0}^{2n-2}\binom{2n-2}{k}[Sin(\pi x)]^{(2n-2-k)}[Tan(\pi x)]^{(k+1)}=$$

$$=\pi\sum_{k=0}^{n-1}\binom{2n-2}{2k}[Sin(\pi x)]^{(2n-2-2k)}[Tan(\pi x)]^{(2k+1)}+\pi\sum_{k=1}^{n}\binom{2n-2}{2k-1}[Sin(\pi x)]^{(2n-1-2k)}[Tan(\pi x)]^{(2k)}$$

Osserviamo che: $[Sin(\pi x)]^{(2n-1-2k)} = [\frac{1}{2i}(e^{i\pi x}-e^{-i\pi x})]^{(2n-1-2k)} =$

$$=[\frac{1}{2i}[(i\pi)^{2n-1-2k}e^{i\pi x}-(-i\pi)^{2n-1-2k}e^{-i\pi x})] = \pi^{2n-1-2k}(-1)^{n-k-1}[\frac{1}{2}[e^{i\pi x}+e^{-i\pi x})] =$$
$$= \pi^{2n-1-2k}(-1)^{n-k-1}Cos(\pi x) \qquad \text{(B.1.3.03)}$$

$$[Sin(\pi x)]^{(2n-2k)} = [\frac{1}{2i}(e^{i\pi x} - e^{-i\pi x})]^{(2n-2k)} = [\frac{1}{2i}[(i\pi)^{2n-2k} e^{i\pi x} - (-i\pi)^{2n-2k} e^{-i\pi x})] =$$

$$= \pi^{2n-2k} (-1)^{n-k} [\frac{1}{2i}[e^{i\pi x} - e^{-i\pi x})] = \pi^{2n-2k} (-1)^{n-k} Sin(\pi x) \qquad \text{(B.1.3.04)}$$

Applicando la (A.1.1.01) e la (A.2.1.01), ricaviamo:

$$(2n-1)! \sum_{k\geq 0} (-1)^k [\frac{1}{(\frac{1}{2} - x + k)^{2n}} - \frac{1}{(\frac{1}{2} + x + k)^{2n}}] =$$

$$\pi^{2n} (-1)^{n-1} \{ \sum_{k=0}^{n-1} \binom{2n-2}{2k} (-1)^k Sin(\pi x) \sum_{h=0}^{k+1} a_h Tan(\pi x)]^{2h} +$$

$$+ \sum_{k=1}^{n} \binom{2n-2}{2k-1} (-1)^k Cos(\pi x) \sum_{h=0}^{k} b_h [Tan(\pi x)]^{2h+1} \} =$$

$$= \pi^{2n} (-1)^{n-1} Sin(\pi x) \{ \sum_{k=0}^{n-1} \binom{2n-2}{2k} (-1)^k \sum_{h=0}^{k+1} a_h Tan(\pi x)]^{2h} +$$

$$+ \sum_{k=1}^{n-1} \binom{2n-2}{2k-1} (-1)^k Cos(\pi x) \sum_{h=0}^{k} b_h [Tan(\pi x)]^{2h} \} \qquad \text{(B.1.3.05)}$$

I coefficienti a_h, b_h sono forniti, rispettivamente, dalle formule (A.1.3.05) e (A.2.3.03).
Per x = ¼, otteniamo:

$$(2n-1)! 4^{2n} \sum_{k\geq 0} (-1)^k [\frac{1}{(1+4k)^{2n}} - \frac{1}{(3+4k)^{2n}}] =$$

$$= \pi^{2n} (-1)^{n-1} \frac{\sqrt{2}}{2} \{ \sum_{k=0}^{n-1} \binom{2n-2}{2k} (-1)^k \sum_{h=0}^{k+1} a_h + \sum_{k=1}^{n} \binom{2n-2}{2k-1} (-1)^k \sum_{h=0}^{k} b_h \}; \qquad \text{(B.1.3.06)}$$

in virtù delle relazioni (A.1.2.01) e (A.2.2.04), troviamo:

$$(2n-1)! 4^{2n} \sum_{k\geq 0} (-1)^k [\frac{1}{(1+4k)^{2n}} - \frac{1}{(3+4k)^{2n}}] =$$

$$= \pi^{2n} (-1)^{n-1} \frac{\sqrt{2}}{2} \{ \sum_{k=0}^{n-1} \binom{2n-2}{2k} (-1)^k 2^{4k+3} (2^{2k+2} - 1) \frac{|B_{2k+2}|}{2k+2} +$$

$$- \sum_{k=1}^{n} \binom{2n-2}{2k-1} 2^{2k} [\frac{1}{2k+1} \sum_{h=1}^{k} \binom{2k+1}{2h} 2^{2h} (2^{2h} - 1) B_{2h} - 1] \} \qquad \text{(B.1.3.07)}$$

B.2.2.- Derivando, (2n) volte, rispetto ad x, la (B.0.02), otteniamo:

$$\int_0^\infty \frac{u^{2n} Cosh(ux)}{Cosh(u/2)} du = [\frac{\pi}{Cos(\pi x)}]^{(2n)} = \qquad |x| < \frac{1}{2}, \qquad \text{(B.2.2.01)}$$

Ponendo, nella (B.2.2.01), x = 0, troviamo:

$$\int_0^\infty \frac{u^{2n}}{Cosh(u/2)} du = \lim_{x \to 0} [\frac{\pi}{Cos(\pi x)}]^{(2n)} = 2\pi \sum_{k \ge 0} (-1)^k (-i\pi)^{2n} (1+2k)^{2n} \ ;$$

$$\int_0^\infty \frac{u^{2n}}{Cosh(u/2)} du = 2 \int_0^\infty u^{2n} e^{-u/2} \sum_{k \ge 0} (-1)^k e^{-uk} du = 2\,(2n)! \sum_{k \ge 0} (-1)^k \frac{1}{(\frac{1}{2}+k)^{2n+1}} =$$

$$= 2^{2n+2} (2n)! \sum_{k \ge 0} (-1)^k \frac{1}{(1+2k)^{2n+1}} \ ;$$

$$2\pi \sum_{k \ge 0} (-1)^k (-i\pi)^{2n} (1+2k)^{2n} = 2\pi^{2n+1} (-1)^n \sum_{k \ge 0} (-1)^k (1+2k)^{2n} =$$

$$= 2\pi^{2n+1} (-1)^n [\frac{1}{2} - \frac{1}{2n+1} \sum_{h \ge 1} \binom{2n+1}{2h} 2^{2h-1} (2^{2h}-1) B_{2h}] \ ;$$

abbiamo utilizzato la (A.2.5.01), per cui:

$$\sum_{k \ge 0} \frac{(-1)^k}{(1+2k)^{2n+1}} = \frac{\pi^{2n+1} (-1)^{n-1}}{2^{2n+2} (2n+1)!} [\sum_{h \ge 1} \binom{2n+1}{2h} 2^{2h} (2^{2h}-1) B_{2h}] - (2n+1)] \qquad \text{(B.2.2.02)}$$

La (B.2.2.02) è perfettamente identica alla (A.2.6.05)

B.2.3 – Ponendo nella (B.2.2.01), x = ¼, otteniamo:

$$\int_0^\infty \frac{u^{2n} Cosh(u/4)}{Cosh(u/2)} du = [\frac{\pi}{Cos(\pi x)}]^{(2n)}_{x=1/4} = 2\pi [\sum_{k \ge 0} (-1)^k e^{-i\pi x(1+2k)}]^{(2n)}_{x=1/4)}$$

Operando, abbiamo:

$$\int_0^\infty \frac{u^{2n} Cosh(u/4)}{Cosh(u/2)} du = \int_0^\infty u^{2n} (e^{u/4} + e^{-u/4}) e^{-u/2} \sum_{k \ge 0} (-1)^k e^{-uk} du =$$

$$= \sum_{k \ge 0} (-1)^k \int_0^\infty u^{2n} (e^{u/4} + e^{-u/4}) e^{-u/2} e^{-uk} du = \sum_{k \ge 0} (-1)^k \int_0^\infty u^{2n} (e^{-u/4} + e^{-3u/4}) e^{-uk} du =$$

$$= (2n)! \sum_{k \ge 0} (-1)^k [\frac{1}{(\frac{1}{4}+k)^{2n+1}} + \frac{1}{(\frac{3}{4}+k)^{2n+1}}] = (2n)! 4^{2n+1} \sum_{k \ge 0} (-1)^k [\frac{1}{(1+4k)^{2n+1}} + \frac{1}{(3+4k)^{2n+1}}] \ ;$$

$$2\pi [\sum_{k \ge 0} (-1)^k e^{-i\pi x(1+2k)}]^{(2n)}_{x=1/4)} = 2\pi \sum_{k \ge 0} (-1)^k (-i\pi)^{2n} (1+2k)^{2n} e^{-i\pi/4} (-i)^k =$$

$$=2\pi^{2n+1}(-1)^n e^{-i\pi/4}\sum_{k\geq 0}(i)^k(1+2k)^{2n} \ ;$$

ora, $e^{-i\pi/4}\sum_{k\geq 0}(i)^k(1+2k)^{2n} = \frac{\sqrt{2}}{2}(1-i)[1+\sum_{k\geq 1}(-1)^k(1+4k)^{2n} - i\sum_{k\geq 1}(-1)^k(4k-1)^{2n}]$;

prendendo le parti reali della relazione precedente, abbiamo:

$$\mathrm{Re}[e^{-i\pi/4}\sum_{k\geq 0}(i)^k(1+2k)^{2n}] = \frac{\sqrt{2}}{2}[1+\sum_{k\geq 1}(-1)^k(1+4k)^{2n} - \sum_{k\geq 1}(-1)^k(1-4k)^{2n}] =$$

$$=\frac{\sqrt{2}}{2}[1+\sum_{k\geq 1}(-1)^k\sum_{h=0}^{2n}\binom{2n}{h}(4k)^h - \sum_{k\geq 1}(-1)^k\sum_{h=0}^{2n}\binom{2n}{h}(-4k)^h] =$$

$$=\frac{\sqrt{2}}{2}[1+\sum_{k\geq 1}(-1)^k\sum_{h=1}^{n}\binom{2n}{2h-1}(4k)^{2h-1} - \sum_{k\geq 1}(-1)^k\sum_{h=0}^{n}\binom{2n}{2h-1}(-4k)^{2h-1}] =$$

$$=\frac{\sqrt{2}}{2}[1+2\sum_{h=1}^{n}\binom{2n}{2h-1}4^{2h-1}\sum_{k\geq 1}(-1)^k k^{2h-1}] =$$

$$=\frac{\sqrt{2}}{2}[1-2\sum_{h=1}^{n}\binom{2n}{2h-1}4^{2h-1}(2^{2h}-1)\frac{B_{2h}}{2h}] = \frac{\sqrt{2}}{2}[1-\frac{1}{2(2n+1)}\sum_{h=1}^{n}\binom{2n+1}{2h}4^{2h}(2^{2h}-1)B_{2h}];$$

$$(2n)!4^{2n+1}\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n+1}}+\frac{1}{(3+4k)^{2n+1}}] = 2\pi^{2n+1}(-1)^n e^{-i\pi/4}\sum_{k\geq 0}(i)^k(1+2k)^{2n} =$$

$$=\pi^{2n+1}(-1)^{n-1}\sqrt{2}[\frac{1}{2(2n+1)}\sum_{h=1}^{n}\binom{2n+1}{2h}4^{2h}(2^{2h}-1)B_{2h}-1] \text{, da cui:}$$

$$\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n+1}}+\frac{1}{(3+4k)^{2n+1}}] =$$

$$=\frac{(-1)^{n-1}\pi^{2n+1}\sqrt{2}}{4^{2n+1}2(2n+1)!}[\sum_{h=1}^{n}\binom{2n+1}{2h}4^{2h}(2^{2h}-1)B_{2h}-2(2n+1)] \qquad \text{(B.2.3.01)}$$

La relazione (B.2.3.01) è stata verificata con un programma di matematica.

B.2.4 Riconsideriamo la (B.2.2.01)

$$\int_0^\infty \frac{u^{2n}Cosh(ux)}{Cosh(u/2)}du = [\frac{\pi}{Cos(\pi x)}]^{(2n)} = 2\pi[\sum_{k\geq 0}(-1)^k e^{-i\pi x(1+2k)}]^{(2n)} \qquad \text{(B.2.4.01)}$$

Operando, abbiamo:

$$\int_0^\infty \frac{u^{2n}Cosh(ux)}{Cosh(u/2)}du = \int_0^\infty u^{2n}(e^{ux}+e^{-ux})\sum_{k\ge 0}(-1)^k e^{-u/2}e^{-uxk}du =$$

$$=(2n)!\sum_{k\ge 0}(-1)^k[\frac{1}{(\frac{1}{2}-x+k)^{2n+1}}+\frac{1}{(\frac{1}{2}+x+k)^{2n+1}}];$$

$$2\pi[\sum_{k\ge 0}(-1)^k e^{-i\pi x(1+2k)}]^{(2n)} = 2\pi[\sum_{k\ge 0}(-1)^k(-i\pi)^{2n}(1+2k)^{2n}e^{-i\pi x(1+2k)}] =$$

$$=2\pi^{2n+1}(-1)^n[\sum_{k\ge 0}(-1)^k(1+2k)^{2n}\sum_{h\ge 0}\frac{(-i\pi x)^h(1+2k)^h}{h!}]$$

Per x reale, la precedente relazione è reale solamente quando h è pari, e quindi:

$$2\pi[\sum_{k\ge 0}(-1)^k e^{-i\pi x(1+2k)}]^{(2n)} = 2\pi^{2n+1}(-1)^n\sum_{h\ge 0}\frac{(-i\pi x)^{2h}}{(2h)!}\sum_{k\ge 0}(-1)^k(1+2k)^{2n+2h} =$$

Applicando la (A.2.5.01), abbiamo:

$$\sum_{k\ge 0}(-1)^k(1+2k)^{2n+2h} = \frac{1}{2}-\frac{1}{2(2n+2h)}\sum_{j=1}^{n+h}\binom{2n+2h+1}{2j}2^{2j}(2^{2j}-1)B_{2j}$$; pertanto:

$$(2n)!\sum_{k\ge 0}(-1)^k[\frac{1}{(\frac{1}{2}-x+k)^{2n+1}}+\frac{1}{(\frac{1}{2}+x+k)^{2n+1}}] =$$

$$=\pi^{2n+1}(-1)^n\sum_{h\ge 0}\frac{(\pi x)^{2h}(-1)^h}{(2h)!}[1-\frac{1}{2n+2h}\sum_{J=1}^{n+h}\binom{2n+2h+1}{2j}2^{2j}(2^{2j}-1)B_{2j}], \qquad (B.2.4.02)$$

Per x = 0, ritroviamo la (B.2.2.02);
per x = ¼, ritroviamo la (B.2.3.01).
Inoltre, operando sulla (B.2.3.01), otteniamo:

$$[\frac{\pi}{Cos(\pi x)}]^{(2n)} = [\frac{\pi}{Cos(\pi x)}]^{(2n-1+1)} = \pi[\frac{\pi Sin(\pi x)}{(Cos(\pi x))^2}]^{(2n-1)} =$$

$$=\pi[Sin(\pi x)DTan(\pi x)]^{(2n-1)} = \pi\sum_{K=0}^{2n-1}\binom{2n-1}{k}[Sin(\pi x)]^{(2n-1-k)}[Tan(\pi x)]^{(k+1)}] =$$

$$=\pi\sum_{K=0}^{n-1}\binom{2n-1}{2k}[Sin(\pi x)]^{(2n-1-2k)}[Tan(\pi x)]^{(2k+1)}]+\pi\sum_{K=1}^{n}\binom{2n-1}{2k-1}[Sin(\pi x)]^{(2n-2k)}[Tan(\pi x)]^{(2k)}];$$

Ricordando le relazioni (B.1.3.03) e (B.1.3.04), abbiamo:

$$[Sin(\pi x)]^{(2n-1-2k)} = \pi^{2n-1-2k}(-1)^{n-k-1}Cos(\pi x) \qquad (B.2.4.03)$$

$$[Sin(\pi x)]^{(2n-2k)} = \pi^{2n-2k}(-1)^{n-k}Sin(\pi x) \qquad (B.2.4.04)$$

Applicando la (A.1.1.01) e la (A.2.1.01), ricaviamo:

$$[\frac{\pi}{Cos(\pi x)}]^{(2n)} = \pi^{2n+1}\sum_{K=0}^{n-1}\binom{2n-1}{2k}(-1)^{n-k-1}Cos(\pi x)\sum_{h=0}^{k+1}a_h Tan(\pi x)]^{2h} +$$

$$+\pi^{2n+1}\sum_{K=1}^{n}\binom{2n-1}{2k-1}(-1)^{n-k}Sin(\pi x)\sum_{h=0}^{k}b_h[Tan(\pi x)]^{2h+1} \qquad (B.2.4.05)$$

I coefficienti a_h, b_h sono forniti, rispettivamente, dalle formule (A.1.3.05) e (A.2.3.03).

Pertanto: $(2n)!\sum_{k\ge 0}(-1)^k[\frac{1}{(\frac{1}{2}-x+k)^{2n+1}}+\frac{1}{(\frac{1}{2}+x+k)^{2n+1}}] =$

$$=\pi^{2n+1}(-1)^{n-1}\{\sum_{K=0}^{n-1}\binom{2n-1}{2k}(-1)^k Cos(\pi x)\sum_{h=0}^{k+1}a_h Tan(\pi x)]^{2h}+$$

$$-\sum_{K=1}^{n}\binom{2n-1}{2k-1}(-1)^k Sin(\pi x)\sum_{h=0}^{k}b_h[Tan(\pi x)]^{2h+1}\};\qquad |x|<\frac{1}{2}, \qquad \text{(B.2.4.06)}$$

per x = 0, abbiamo: $(2n)!2^{2n+2}\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}}=\pi^{2n+1}(-1)^{n-1}\{\sum_{K=0}^{n-1}\binom{2n-1}{2k}(-1)^k a_0=$

$$=\pi^{2n+1}(-1)^{n-1}\sum_{K=0}^{n-1}\binom{2n-1}{2k}(-1)^k 2^{2k+2}(2^{2k+2}-1)\frac{|B_{2k+2}|}{2k+2};\qquad \text{(B.2.4.07)}$$

Per x = ¼, otteniamo:

$$(2n)!4^{2n+1}\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n+1}}+\frac{1}{(3+4k)^{2n+1}}]=\pi^{2n+1}(-1)^{n-1}\frac{\sqrt{2}}{2}\{\sum_{K=0}^{n-1}\binom{2n-1}{2k}(-1)^k\sum_{h=0}^{k+1}a_h+$$

$$-\sum_{K=1}^{n}\binom{2n-1}{2k-1}(-1)^k\sum_{h=0}^{k}b_h\}$$

C.0.0- Utilizzando la nota relazione

$$\int_0^\infty \frac{t^{z-1}dt}{1+t}=\frac{\pi}{Sin\pi z},\ 0<z<1,$$

e ponendo z = 1/n, con n > 1, otteniamo:

$$\int_0^\infty \frac{t^{\frac{1}{n}-1}dt}{1+t}=\frac{\pi}{Sin\frac{\pi}{n}}=(t=u^n)=\int_0^\infty \frac{u^{n(\frac{1}{n}-1)}nu^{n-1}du}{1+u^n}=n\int_0^\infty\frac{du}{1+u^n}=$$

$$=n[\int_0^1\frac{du}{1+u^n}+\int_0^1\frac{u^{n-2}du}{1+u^n}]=n\int_0^1\frac{1+u^{n-2}du}{1+u^n}.$$

Ora, $\frac{n(1+u^{n-2})}{1+u^n}=n\sum_{k=0}^{n-1}\frac{1+u_k^{n-2}}{u-u_k}\frac{1}{nu_k^{n-1}}=\sum_{k=0}^{n-1}\frac{1+u_k^{n-2}}{u-u_k}\frac{u_k}{u_k^n}$,

dove $u_k^n=-1$, $u_k=e^{i\pi(2k+1)/n}$, k = 0,1,2,3,…(n-1); quindi:

$\frac{n(1+u^{n-2})}{1+u^n}=-\sum_{k=0}^{n-1}(u_k-\frac{1}{u_k})\frac{1}{u-u_k}$; pertanto:

$$\int_0^\infty\frac{t^{\frac{1}{n}-1}dt}{1+t}=\frac{\pi}{Sin\frac{\pi}{n}}=-\sum_{k=0}^{n-1}(u_k-\frac{1}{u_k})[\ln(u-u_k]_0^1=-\sum_{k=0}^{n-1}(u_k-\frac{1}{u_k})[\ln(1-\frac{1}{u_k})=$$

$$=-\sum_{k=0}^{n-1}(e^{i\pi(2k+1)/n}-e^{-i\pi(2k+1)/n})\ln(1-e^{-i\pi(2k+1)/n});\qquad \text{(C.0.0.01)}$$

$$\ln[1-e^{-i\pi(2k+1)/n}]=\ln\{1-Cos[\pi(2k+1)/n]+iSin[\pi(2n+1)/n]\}=$$

$$=\frac{1}{2}\ln\{(1-Cos[\pi(2k+1)/n])^2+(Sin[\pi(2n+1)/n])^2\}+iArcTan\frac{Sin[\pi(2k+1)/n]}{1-Cos[\pi(2k+1)/n]};$$

$$iArcTan\frac{Sin[\pi(2k+1)/n]}{1-Cos[\pi(2k+1)/n]}=iArcTan\frac{Cos[\pi(2k+1)/(2n)]}{Sin[\pi(2k+1)/(2n)]}=$$
$$=iArcTan\frac{Sin\{(\pi/2)[1-(2k+1)/n]\}}{Cos\{(\pi/2)[1-(2k+1)/n]\}}=i\frac{\pi}{2}(1-\frac{2k+1}{n}).$$

Sostituendo nella (C.0.0.01), ed uguagliando le parti reali, ricaviamo:

$$\int_0^\infty \frac{t^{\frac{1}{n}-1}dt}{1+t}=\frac{\pi}{Sin\frac{\pi}{n}}=\sum_{k=0}^{n-1}2Sin[\pi(2k+1)/n]\{\frac{\pi}{2}(1-\frac{2k+1}{n})\}=$$
$$=\pi\sum_{k=0}^{n-1}(\frac{n-2k-1}{n})Sin[\pi(2k+1)/n],$$ cioè:
$$\sum_{k=0}^{n-1}(n-2k-1)Sin[\pi(2k+1)/n]Sin\frac{\pi}{n}=n$$

Ricordando che : Cos(a+b) - Cos(a-b) = -2Sin(a)Sin(b), otteniamo:

$$\sum_{k=0}^{n-1}(n-2k-1)(-\frac{1}{2})[Cos\frac{2\pi(k+1)}{n}-Cos\frac{2\pi k}{n}]=n;$$

sostituendo, nella precedente relazione, k a k+1, troviamo:

$$\sum_{k=1}^{n}(n-2k+1)[Cos\frac{2\pi(k-1)}{n}-Cos\frac{2\pi k}{n}]=2n; \qquad \text{(C.0.0.02)}$$

Inoltre, osserviamo che:

$\sum_{k=1}^{n}[Cos\frac{2\pi(k-1)}{n}-Cos\frac{2\pi k}{n}]=0$, e quindi dalla (C.0.0.02), ricaviamo:

$\sum_{k=1}^{n}(-2k)[Cos\frac{2\pi(k-1)}{n}-Cos\frac{2\pi k}{n}]=2n$, cioè:

$$\sum_{k=1}^{n}k[Cos\frac{2\pi k}{n}-Cos\frac{2\pi(k-1)}{n}]=n \qquad \text{(C.0.0.03)}$$

La relazione (C.0.0.03) è stata verificata con un programma di matematica.

Riteniamo utile riportare in un unico schema le interessanti formule trovate.

1) P(x) = Tan(x), $[Tan(x)]^{(2n-1)} = \sum_{h=0}^{n}a_h[Tan(x)]^{2h}$

$$a_0 = 2^{2n}(-1)^{n-1}(2^{2n}-1)\frac{B_{2n}}{2n} = 2^{2n}(2^{2n}-1)\frac{|B_{2n}|}{2n}, \qquad \text{(A.1.1.06)}$$

2) $$\sum_{k\geq 0}\frac{1}{(1+2k)^{2n}} = \frac{(2^{2n}-1)\pi^{2n}}{2(2n)!}|B_{2n}| \qquad \text{(A.1.1.10)}$$

3) $$\sum_{h=0}^{n}a_h = 2^{4n-1}(2^{2n}-1)\frac{|B_{2n}|}{2n}, \qquad \text{(A.1.2.01)}$$

4) $$(\sum_{h=0}^{n} a_h)/a_0 = 2^{2n-1} \qquad \text{(A.1.2.02)}$$

5) $$a_h = \frac{2^{2n}}{(2h)!}(-1)^{n-1}(-1)^h[\sum_{u=0}^{h} s(2h,2u)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u} +$$
$$+2h\sum_{u=1}^{h} s(2h-1,2u-1)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u} +$$
$$+\sum_{j=2}^{2h}\binom{2h}{j}\sum_{u=0}^{2h-1} s(2h-1,u)\sum_{p=1}^{[(u+1)/2]}\binom{u}{2p-1}(j-1)^{u-2p+1}(2^{2n+2p}-1)\frac{B_{2n+2p}}{2n+2p}]. \qquad \text{(A.1.3.05)}$$

6) $$a_n = -\frac{2^{2n}}{(2n)!}[\sum_{u=0}^{n} s(2n,2u)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u} +$$
$$+2n\sum_{u=1}^{n} s(2n-1,2u-1)(2^{2n+2u}-1)\frac{B_{2n+2u}}{2n+2u} +$$
$$+\sum_{j=2}^{2n}\binom{2n}{j}\sum_{u=0}^{2n-1} s(2n-1,u)\sum_{p=1}^{[u/2]}\binom{u}{2p-1}(j-1)^{u-2p+1}(2^{2n+2p}-1)\frac{B_{2n+2p}}{2n+2p}] = \mathbf{(2n-1)!}, \qquad \text{(A.1.3.06)}$$

7) $$\sum_{k\geq 0}[\frac{1}{(\frac{1}{2}-x+k)^{2n}} + \frac{1}{(\frac{1}{2}+x+k)^{2n}}] = \frac{2^{2n}}{(2n-1)!}\pi^{2n}\sum_{h\geq 0}\frac{(2\pi x)^{2h}}{(2h)!}\frac{(2^{2n+2h}-1)|B_{2n+2h}|}{2n+2h} =$$

$$= \frac{\pi^{2n}}{(2n-1)!}\sum_{h=0}^{n} a_h [Tan(\pi x)]^{2h} \qquad \text{(A.1.5.07)}$$

(8) $$[Tan(x)]^{(2n)} = \sum_{h=0}^{n} b_h [Tan(x)]^{2h+1}$$

$$b_0 = \frac{2^{2n+1}}{n+1}(2^{2n+2}-1)|B_{2n+2}| \qquad \text{(A.2.1.03)}$$

9) $$\sum_{k\geq 1}(-1)^k(2k-1)^{2n} = -\frac{1}{2} + \frac{1}{2(2n+1)}\sum_{h\geq 1}\binom{2n+1}{2h}(-2)^{2h}(2^{2h}-1)B_{2h}, \qquad \text{(A.2.2.03)}$$

10) $$\sum_{h=0}^{n} b_h = 2^{2n}(-1)^{n-1}[\frac{1}{2n+1}\sum_{h=1}^{n}\binom{2n+1}{2h}2^{2h}(2^{2h}-1)B_{2h} - 1] \qquad \text{(A.2.2.04)}$$

11) $$b_h = \frac{2^{2n+1}}{(2h+1)!}(-1)^n(-1)^h[\sum_{u=0}^{h} s(2h+1,2u+1)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2} +$$

$$+(2h+1)\sum_{u=0}^{h} s(2h,2u)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2} +$$

$$+\sum_{j=2}^{2h+1}\binom{2h+1}{j}\sum_{u=0}^{2h} s(2h,u)\sum_{p=0}^{[u/2]}\binom{u}{2p}(j-1)^{u-2p}(2^{2n+2p+2}-1)\frac{B_{2n+2p+2}}{2n+2p+2}] \qquad \text{(A.2.3.03)}$$

12) $$b_n=\frac{2^{2n+1}}{(2n+1)!}[\sum_{u=0}^{n}s(2n+1,2u+1)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}+$$

$$+(2n+1)\sum_{u=0}^{n}s(2n,2u)(2^{2n+2u+2}-1)\frac{B_{2n+2u+2}}{2n+2u+2}$$

$$+\sum_{j=2}^{2n+1}\binom{2n+1}{j}\sum_{u=0}^{2n}s(2n,u)\sum_{p=0}^{[u/2]}\binom{u}{2p}(j-1)^{u-2p}(2^{2n+2p+2}-1)\frac{B_{2n+2p+2}}{2n+2p+2}] = \mathbf{(2n)!} \qquad \text{(A.2.3.04)}$$

13) $$\frac{2}{2n+1}\sum_{h=1}^{n}\binom{2n+1}{2h}(2^{2h}-1)B_{2h}=1 \qquad \text{(A.2.4.02)}$$

14) $$\sum_{k\geq 0}(-1)^k(1+2k)^{2n}=\frac{1}{2}-\frac{1}{2(2n+1)}\sum_{h\geq 1}\binom{2n+1}{2h}2^{2h}(2^{2h}-1)B_{2h}; \qquad \text{(A.2.5.01)}$$

15) $$\sum_{k\geq 0}\frac{(-1)^k}{(1+2k)^{2n+1}}=\frac{\pi^{2n+1}(-1)^{n-1}}{2^{2n+2}(2n+1)!}[\sum_{h=1}^{n}\binom{2n+1}{2h}2^{2h}(2^{2h}-1)B_{2h}-(2n+1)] \qquad \text{(A.2.5.02)}$$

16) $$\sum_{k\geq 0}[\frac{1}{(1+4k)^{2n+1}}-\frac{1}{(3+4k)^{2n+1}}]=\frac{1}{(2n)!4^{2n+1}}\sum_{h=0}^{n}b_h=$$

$$=\frac{(2\pi)^{2n+1}}{(2n)!4^{2n+1}}\sum_{h\geq 1}\frac{(\pi/2)^{2h-1}}{(2h-1)!}\frac{(2^{2n+2h}-1)|B_{2n+2h}|}{2n+2h} \qquad \text{(A.2.6.05)}$$

17) $$\sum_{k\geq 0}(-1)^k(1+2k)^{2n-1}=0 \qquad \text{(B.1.1.02)}$$

18) $$\frac{1}{2n}\sum_{h=1}^{n}\binom{2n}{2h}2^{2h}(2^{2h}-1)B_{2h}=1 \qquad \text{(B.1.1.03)}$$

19) $$\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n}}-\frac{1}{(3+4k)^{2n}}]=$$

$$=\pi^{2n}(-1)^{n-1}\sqrt{2}\frac{4^{-2n}}{2(2n)!}[\sum_{h=1}^{n}\binom{2n}{2h}4^{2h}(2^{2h}-1)B_{2h}-4n]. \qquad \text{(B.1.2.01)}$$

20) $$(2n-1)!\sum_{k\geq 0}(-1)^k[\frac{1}{(\frac{1}{2}-x+k)^{2n}}-\frac{1}{(\frac{1}{2}+x+k)^{2n}}]=$$

$$=\pi^{2n}(-1)^{n-1}Sin(\pi x)\{\sum_{k=0}^{n-1}\binom{2n-2}{2k}(-1)^k\sum_{h=0}^{k+1}a_h Tan(\pi x)]^{2h}+$$

$$+\sum_{k=1}^{n-1}\binom{2n-2}{2k-1}(-1)^k\sum_{h=0}^{k}b_h[Tan(\pi x)]^{2h}\} \qquad \text{(B.1.3.05)}$$

21) $\sum_{k\geq 0}(-1)^k[\frac{1}{(1+4k)^{2n+1}}+\frac{1}{(3+4k)^{2n+1}}]=$

$$=\frac{(-1)^{n-1}\pi^{2n+1}\sqrt{2}}{4^{2n+1}2(2n+1)!}[\sum_{h=1}^{n}\binom{2n+1}{2h}4^{2h}(2^{2h}-1)B_{2h}-2(2n+1)] \qquad \text{(B.2.3.01)}$$

22) $$(2n)!\sum_{k\geq 0}(-1)^k[\frac{1}{(\frac{1}{2}-x+k)^{2n+1}}+\frac{1}{(\frac{1}{2}+x+k)^{2n+1}}] =$$

$$=\pi^{2n+1}(-1)^n\sum_{h\geq 0}\frac{(\pi x)^{2h}(-1)^h}{(2h)!}[1-\frac{1}{2n+2h}\sum_{J=1}^{n+h}\binom{2n+2h+1}{2j}2^{2j}(2^{2j}-1)B_{2j}], \qquad \text{(B.2.4.02)}$$

23) $$(2n)!\sum_{k\geq 0}(-1)^k[\frac{1}{(\frac{1}{2}-x+k)^{2n+1}}+\frac{1}{(\frac{1}{2}+x+k)^{2n+1}}]=$$

$$=\pi^{2n+1}(-1)^{n-1}\{\sum_{K=0}^{n-1}\binom{2n-1}{2k}(-1)^k Cos(\pi x)\sum_{h=0}^{k+1}a_h Tan(\pi x)]^{2h}+$$

$$-\sum_{K=1}^{n}\binom{2n-1}{2k-1}(-1)^k Sin(\pi x)\sum_{h=0}^{k}b_h[Tan(\pi x)]^{2h+1}\}; \qquad |x|<\frac{1}{2}, \qquad \text{(B.2.4.06)}$$

24) $$\sum_{k=1}^{n}k[Cos\frac{2\pi k}{n}-Cos\frac{2\pi(k-1)}{n}]=n \qquad \text{(C.0.0.03)}$$

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Dott. Ing. Pasquale Cutolo

Riferimenti

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